WHAT DOES IT TAKE TO CERTIFY CONVERSION?

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FSCD 2025



Department of Computer Science and Technology What am I trying to do?

We keep telling the world they should verify their critical code...

Yet we still don't have verified kernels!*

*For dependent types: CANDLE exists and it's really cool

Yet we still don't have verified kernels!*

The programs are (relatively) simple...

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The programs are (relatively) simple...

But the reasons why they work are very complicated!

- METAROCQ: ROCQ in ROCQ
- Martin-Löf à la Coq: a place to experiment

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Key characteristics:

- We care about the actual implementation
- We need to manipulate extrinsically typed terms
- We are fundamentally limited by Gödel's 2nd incompleteness theorem

Logical relations

Irrelevant computational content



Models

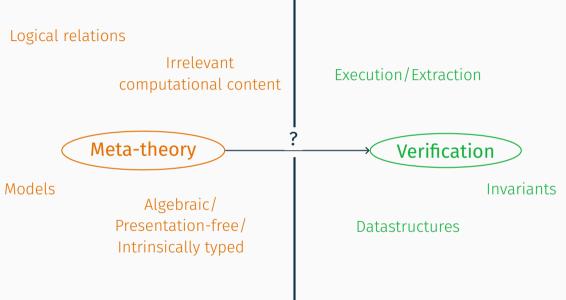
Algebraic/ Presentation-free/ Intrinsically typed

Execution/Extraction



Invariants

Datastructures



THE ALGORITHMS (AND THEIR SPECIFICATION)

Specification

```
Rules for each term/type former (\Pi, \Sigma, Id, \mathcal{U}, \mathbb{N}, \bot ...) +
```

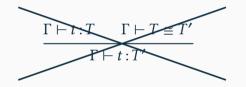
$$\frac{\Gamma \vdash t: T \qquad \Gamma \vdash T \cong T'}{\Gamma \vdash t: T'}$$

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$$\frac{\Gamma \vdash t: T \qquad \Gamma \vdash T \cong T'}{\Gamma \vdash t: T'}$$

Algorithm



Bidirectional: rules for each term former integrate (some) conversion.

Declarative specification

Arbitrarily mixing:

- Refl./Sym./Trans.
- Congruences
- Computation (β)
- Extensionality (η)

Typed!

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Alternate

- 1. β -reduction to whnf
- 2. Type-directed η
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- + closer to specification
- + supports fancier rules
- slower

- + faster
- + simpler (?)
- further from spec.

THE PLAN

WHAT'S IN A DECISION PROCEDURE?

$$P: D \to \mathbb{P} \qquad \stackrel{?}{\Leftrightarrow} \qquad p: D \to \mathbb{B}$$

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- **1.** soundness: $p d = \text{true} \Rightarrow P d$
- 2. completeness: $P d \Rightarrow p d =$ true
- 3. profit!

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How do you know that the type-checker terminates?

- 1. soundness: $p \ d = \text{true} \Rightarrow P \ d$ Look at the trace of the type-checker
- 2. completeness: $P \ d \Rightarrow p \ d =$ true reflexivity \simeq normalisation
- 3. profit?

$$P: D \to \mathbb{P} \qquad \stackrel{?}{\Leftrightarrow} \qquad p: D \to \mathbb{B}$$

1. positive soundness: $p \ d = \text{true} \Rightarrow P \ d$ Look at the trace of the type-checker

$$P: D \to \mathbb{P} \quad \stackrel{?}{\Leftrightarrow} \quad p: D \to \mathbb{B}$$

- **1.** positive soundness: $p d = \text{true} \Rightarrow P d$ Look at the trace of the type-checker
- 2. negative soundness: $p \ d = \text{false} \Rightarrow \neg (P \ d)$ Look (harder) at the trace of the type-checker

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- 3. termination: $(p \ d = \text{true}) \lor (p \ d = \text{false})$ Still hard, of course...

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- 3. termination: $(p \ d = \text{true}) \lor (p \ d = \text{false})$ Still hard, of course...

Now we have a plan

What meta-theory do we need?

THE GOOD PROPERTIES

Injectivity and no-confusion of type constructors

If $\Gamma \vdash T \cong T'$ and T, T' are weak-head normal form, then:

- $T = \mathbb{N} = T'$
- or $T = \prod x$: A.B, $T' = \prod x$: A'.B', with $\Gamma \vdash A' \cong A$ and Γ, x : $A' \vdash B \cong B'$
- or ...
- or $T,\,T'$ are both neutral, and $\Gamma \vdash T \cong T': \mathcal{U}$

Any non-diagonal case is impossible (no-confusion).

THE GOOD PROPERTIES

Injectivity and no-confusion of type constructors

Injectivity and no-confusion at $\ensuremath{\mathbb{N}}$

If $\Gamma \vdash n \cong n': \mathbb{N}$ and n, n' are weak-head normal forms, then:

- n = 0 = n'
- or $n = \mathrm{S}(t), n' = \mathrm{S}(t')$, with $\Gamma \vdash t \cong t' : \mathbb{N}$
- or *n*, *n*′ are both neutral.

Injectivity and no-confusion of type constructors

Injectivity and no-confusion at $\mathbb N$

Injectivity and no-confusion at $\ensuremath{\mathcal{U}}$

•••

THE GOOD PROPERTIES

Injectivity and no-confusion of type constructors

Injectivity and no-confusion at $\mathbb N$

Injectivity and no-confusion at $\ensuremath{\mathcal{U}}$

Injectivity of neutral eliminators^{*}

If $\Gamma \vdash n \cong n': T$ and n and n' are neutrals, then

•
$$n = x = n^{\prime}$$

• or
$$n=m$$
 $u,n'=m'$ u' with $m\cong m'$ and $u\cong u'$

• or $n = \operatorname{rec}_{\mathbb{N}}(m, x.P, t_0, x.y.t_S)$, $n' = \operatorname{rec}_{\mathbb{N}}(m', x.P', t_0', x.y.t_S')$, and ...

See paper/talk to me for subtleties.

Injectivity and no-confusion of type constructors

Injectivity and no-confusion at $\mathbb N$

Injectivity and no-confusion at $\ensuremath{\mathcal{U}}$

Injectivity of neutral eliminators*

Deep normalisation

Every well-typed term is deeply normalising at its type. Every well-formed type is deeply normalising.

	Positive soundness	Negative soundness (typed conversion)	Negative soundness (untyped conversion)	Termination
Injectivity of type constructors	×	×	×	×
Term-level injectivities		$ imes^*$	$ imes^{\star}$	
Normalisation				×

* Not *quite* the same for neutrals

	Positive soundness	Negative soundness (typed conversion)	Negative soundness (untyped conversion)	Termination
Injectivity of type constructors	×	× Roo	a ×	×
Term-level injectivities		(Ap)F	proved ×*	
Normalisation				×

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Injectivity of type constructors	×	× Roo	a ×	×
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Normalisation				×

Injectivities are the important properties

	Positive soundness	Negative soundness (typed conversion)	Negative soundness (untyped conversion)	Termination
Injectivity of type constructors	×	× Roo	a ×	×
Term-level injectivities		(Ap)F	proved ×*	
Normalisation				×

Injectivities are the important properties

Claim/conjecture: this analysis scales to realistic proof assistant kernels

How to prove the properties?

	Logical relation [AÖV17; Adj+24]	Rewriting/Confluence [Tak95]/METAROCQ	Gluing/Nf Model [Ste21; BKS23]	Domain model [CH18]
Syntax	Raw	Raw	Intrinsic	Raw (Intrinsic?)
Weak ambiant theory	×	\checkmark	×	\checkmark
Normalisation	1	×	\checkmark	×
Scaling	×	\checkmark	?	?
η laws	1	×	\checkmark	\checkmark
	Most explored	Insane scaling	Formalisation currently difficult	Very unexplored

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Syntax	Raw	Raw	Intrinsic	Raw (Intrinsic?)
Weak ambiant theory	×	\checkmark	×	1
Normalisation	\checkmark	There is space fo	r exploration!	×
Scaling	×	·		?
η laws	\checkmark	×	\checkmark	\checkmark
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Normalisation	\checkmark	There is space fo	r exploration!	×
Scaling	×	·		?
η laws	1	but I have peculiar requirements.		1
	Most explored	Insane scaling	currently difficult	Very unexplored 11/12

- You can (should!) separate meta-theory and implementation
- Injectivity properties are key, more so that normalisation
- You cannot beat Gödel, but you can salvage a lot with negative soundness
- There is space for new proof/formalisation techniques for meta-theory

You can (should!) separate meta-theory and implementation Injectivity properties are key, more so that normalisation You cannot beat Gödel, but you can salvage a lot with negative soundness There is space for new proof/formalisation techniques

THANK YOU!

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