# Formalised meta-theory for a verified type-theoretic kernel

GdT Malinca – October 22nd 2025

Meven Lennon-Bertrand


ROCQ

Agda


LEAN

Isabelle
HOL





Great successes!





**Great successes!**
**But what makes them usable?**





Pattern-matching

(Strong) records

(Computational) univalence

Termination checking

(Co)Inductive types

Universes

Proof irrelevance

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Proof irrelevance

Gradual typing

Modalities

Observational equality

Subtyping

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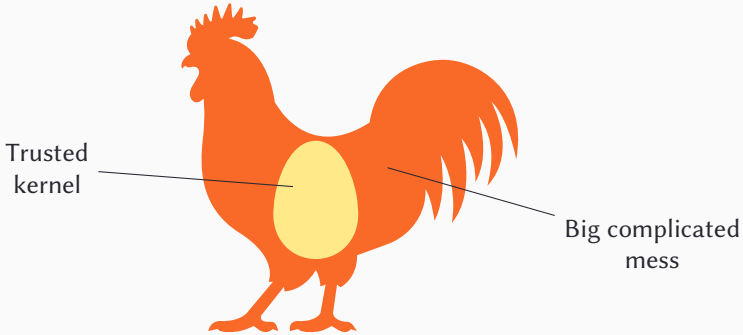
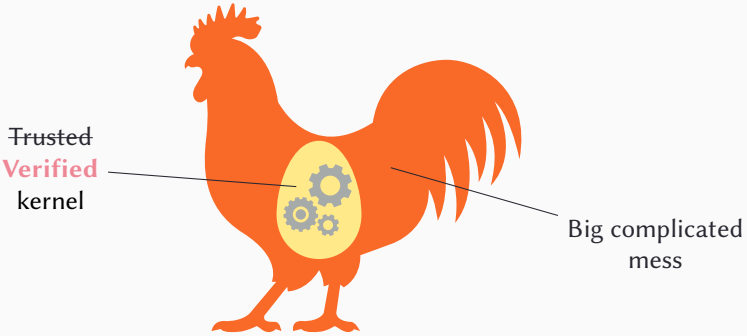
**Real proof assistants are complicated**




The de Bruijn architecture




The de Bruijn architecture: a perfect target for **verification**

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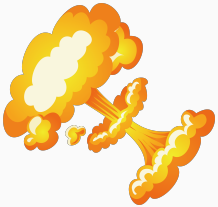
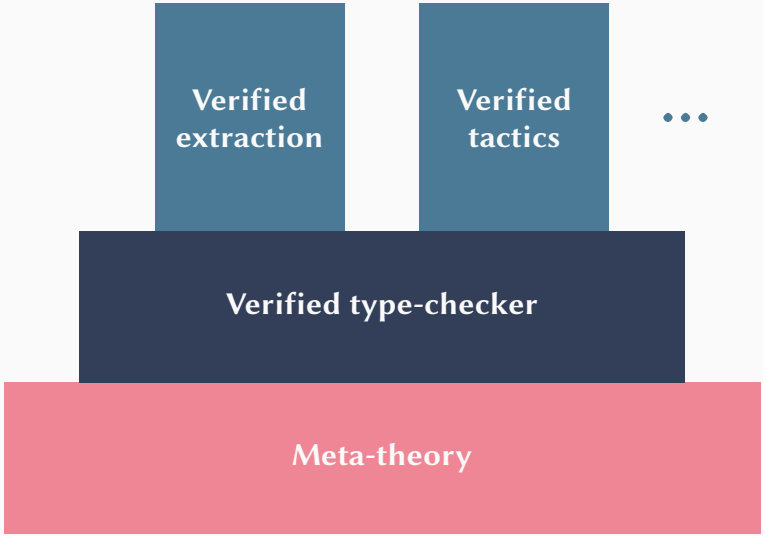
Dependent type theory + Invariants

=




Verified type-checker
Meta-theory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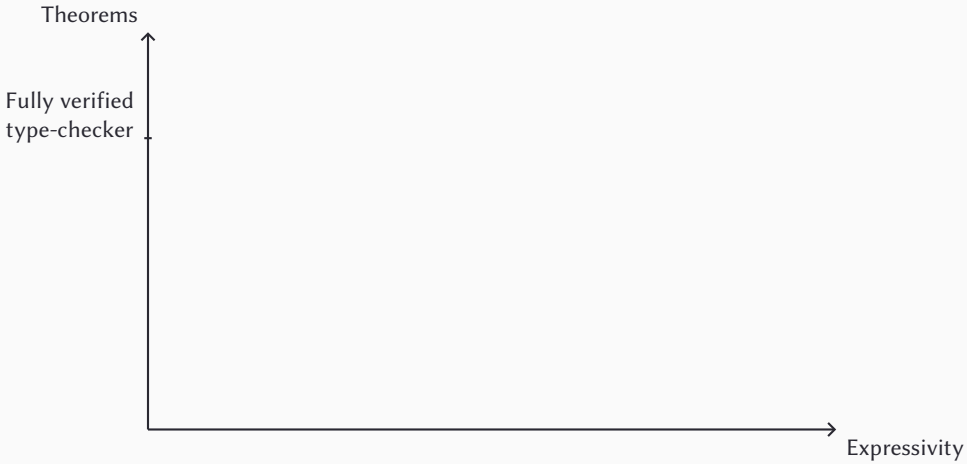
Theorems
Fully verified
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Expressivity

# Expressivity?

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**Proof-theoretic strength does not measure expressivity**

Rocq in Rocq?

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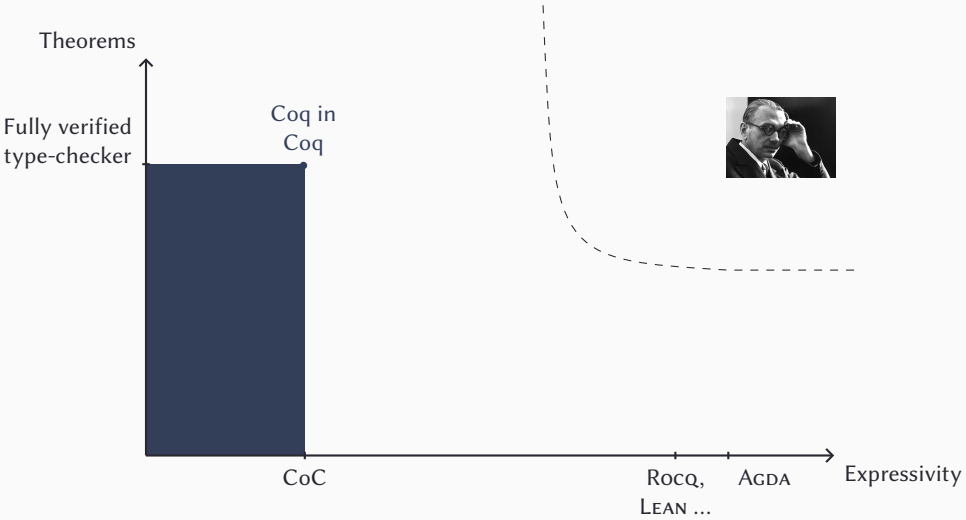
~~Rocq in Rocq?~~

The meta-theory of an object theory $\mathcal{T}$ in a **(slightly) stronger ambient theory** $\mathcal{T}'$.
Or: **almost all** the meta-theory of a theory $\mathcal{T}$ in $\mathcal{T}$ itself.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Rocq, Lean ...
Agda
Expressivity

A long shopping list:

- substitution for term variables
- substitution for universe variables
- weakening
- strengthening
- injectivity of type constructors
- confluence
- standardisation
- safety
- progress
- preservation/subject reduction
- uniqueness of types
- canonicity
- logical consistency
- normalisation
- decidability

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**What properties are needed to verify what?**

**How do we prove these properties?**

# Dependent type theory

# What's type theory anyway?


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- a system of rules/a language
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  - valid logical/mathematical inferences
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} Curry-Howard: **These are essentially the same!**

# A system of rules

$$\frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B}$$

$$\frac{\Gamma, x{:}\,A \vdash t : B}{\Gamma \vdash \lambda\, x{:}\,A.t : A \to B}$$

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Formally read as:

- inductively defined predicates on tree-like structures
- directly defining an algebraic structure

Dependent types:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathrm{Vect}\, A\, n}$$

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The real beast is **conversion**/**definitional equality**:

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$$\frac{\Gamma \vdash [1; 2; 7] : \mathrm{Vect}\, \mathbb{N}\, (1 + 1 + 1) \qquad \Gamma \vdash \mathrm{Vect}\, \mathbb{N}\, (1 + 1 + 1) \cong \mathrm{Vect}\, \mathbb{N}\, 3}{\Gamma \vdash [1; 2; 7] : \mathrm{Vect}\, \mathbb{N}\, 3}$$

# Definitional equality

The least **equivalence relation**

$$\textsc{Refl}\ \frac{\Gamma \vdash t : A}{\Gamma \vdash t \cong t : A} \qquad \textsc{Sym}\ \frac{\Gamma \vdash t \cong u : A}{\Gamma \vdash u \cong t : A} \qquad \textsc{Trans}\ \frac{\Gamma \vdash t \cong u : A \qquad \Gamma \vdash u \cong v : A}{\Gamma \vdash t \cong v : A}$$

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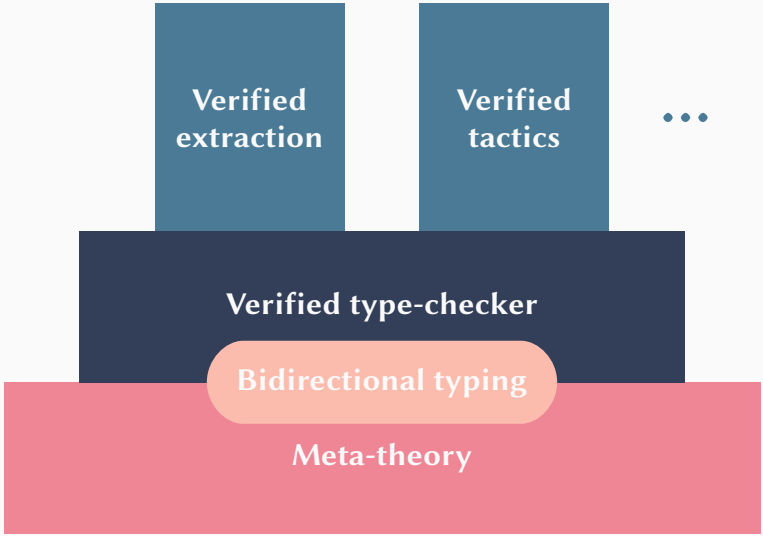
If we orient β-rules, we get **reduction** $\to$.

# Bidirectional typing


Verified extraction
Verified tactics
...
Verified type-checker
Bidirectional typing
Meta-theory

# Information Must Flow

**Inference and checking**

$\Gamma \vdash t : A$ separates into

inference: $\Gamma \vdash t \triangleright A$ checking: $\Gamma \vdash t \triangleleft A$

Similar “meaning”, different modes: input/subject/output.

# Structure!

$$\frac{\vdash \Gamma \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}$$

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- Globally enforce well-formation

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- ~~Globally enforce well-formation~~ → **Well-formation as an invariant**
- Clear **information flow**

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- ~~Globally enforce well-formation~~ → Well-formation as an invariant
- Clear information flow
- ~~Unconstrained conversion~~ → Computation ($\to^{\star}$ or $\cong$) **guided by the mode**

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**McBride:** ***A rule is a server for its conclusion and a client for its premises.***

- Modes guide invariant preservation
- In a conclusion, you assume inputs are well-formed, and ensure outputs are
- In a premise, you ensure inputs are well-formed, and assume outputs are

# Specifying bidirectional typing

**Positive soundness**

If the algorithm answers **yes** (*i.e.* $\Gamma \vdash t \triangleright T$) and its preconditions are met, then $\Gamma \vdash t : T$.

**Negative soundness**

If the algorithm answers **no** (*i.e.* $\Gamma \vdash t \triangleright \lightning$) and its preconditions are met, then $t$ is not typable.

**Totality**

If its preconditions are met, the algorithm always answers.

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Positive soundness + Negative soundness + Totality $\Longrightarrow$

**Completeness** the answer is always yes on typable terms

**Decidability**

# MetaRocq

MetaCoq is developed by (left to right) Abhishek Anand, Danil Annenkov, Simon Boulier, Cyril Cohen, Yannick Forster, Jason Gross, Meven Lennon-Bertrand, Kenji Maillard, Gregory Malecha, Jakob Botsch Nielsen, Matthieu Sozeau, Nicolas Tabareau and Théo Winterhalter.

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A dependent type theory with

- Very general (co-)inductive types
- Pattern-matching and fixed-points
- Complex universes + cumulativity

} What real users need!

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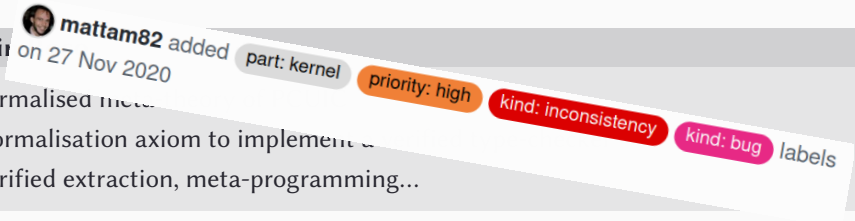

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**Rocq, in Rocq**

- Formalised meta-theory of PCUIC
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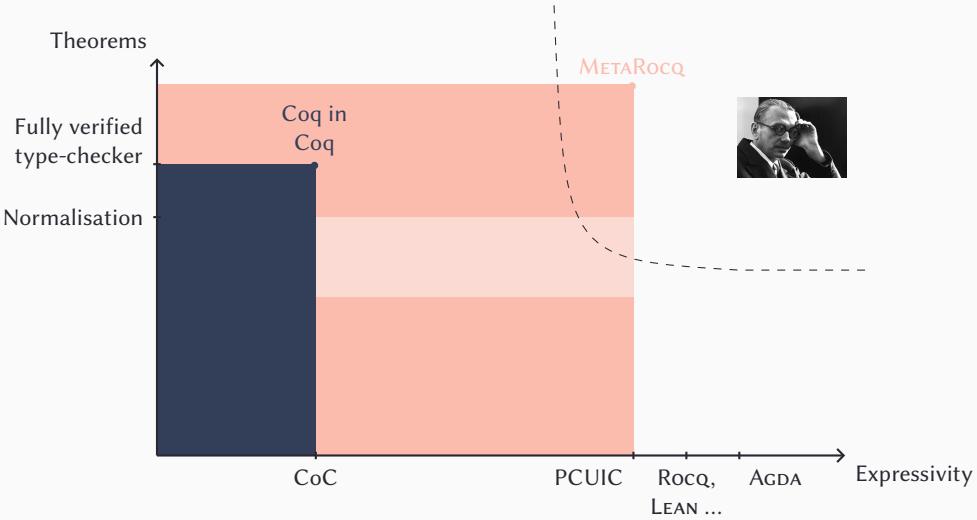
**Evaluation terminates**. Consequences:

- totality ♥
- logical consistency ⚠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# The $1000 question: inversion

If you know $T \cong T'$, **what can you conclude**?

- $T \cong \Pi\, x{:}\, A.B \overset{?}{\Rightarrow} T \to^\star \Pi\, x{:}\, A'.B'$?
- $\Pi\, x{:}\, A.B \cong \Pi\, x{:}\, A'.B' \overset{?}{\Rightarrow} A \cong A' \wedge B \cong B'$?
- $\Pi\, x{:}\, A.B \cong \mathbb{N} \overset{?}{\Rightarrow} \bot$?
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(after Tait, Martin-Löf, Takahashi)

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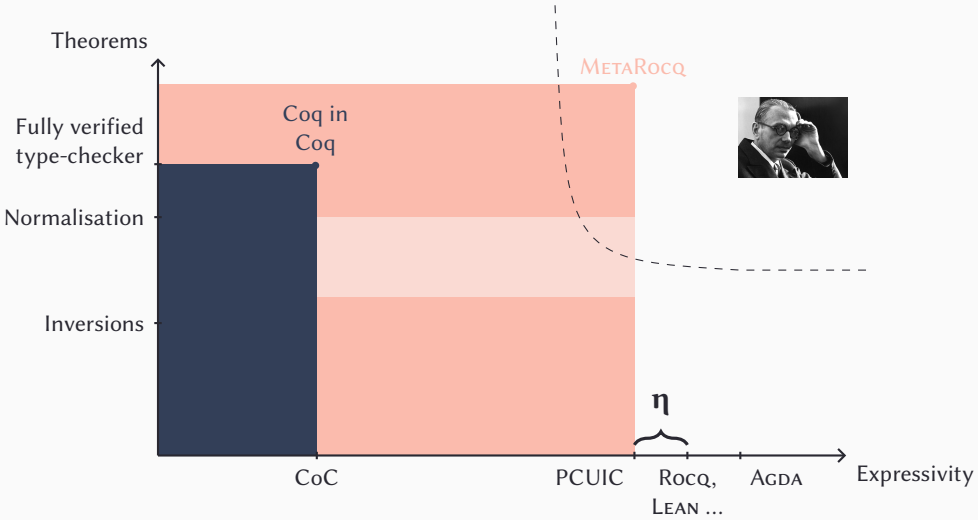
**Confluence**[*]

(after Tait, Martin-Löf, Takahashi)

[*] **Only works** for a **rather poor** notion of conversion...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# Martin-Löf *à la* Coq

Arthur Adjedj

Kenji Maillard

Pierre-Marie Pédrot

Loïc Pujet

$$\eta\textsc{Fun}\ \frac{\Gamma \vdash f : \Pi\, x{:}\, A.B}{\Gamma \vdash f \cong \lambda x{:}\, A.f\ x : \Pi\, x{:}\, A.B}$$

# Extensionality rules

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Might look easy (?), but:

- crucial in practice
- does not really work with confluence
- a very typed notion

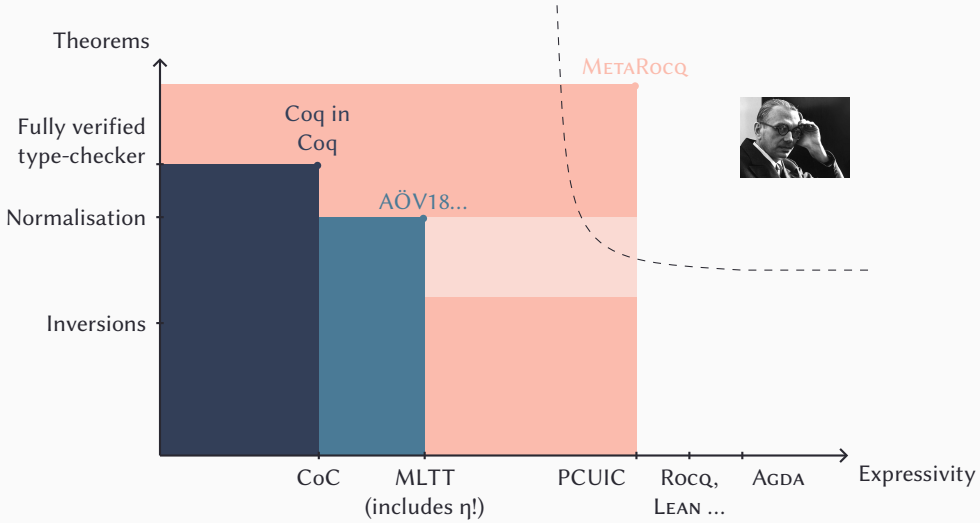
# Extensionality rules

$$\eta\textsc{Fun}\ \frac{\Gamma \vdash f : \Pi\, x{:}\, A.B}{\Gamma \vdash f \cong \lambda x{:}\, A.f\ x : \Pi\, x{:}\, A.B} \qquad \eta\textsc{Prop}\ \frac{\Gamma \vdash P : \mathrm{SProp} \qquad \Gamma \vdash p : P \qquad \Gamma \vdash p' : P}{\Gamma \vdash p \cong p' : P}$$

Might look easy (?), but:

- crucial in practice
- does not really work with confluence
- a very typed notion

Rocq has a clever **untyped implementation**, but what's its **specification**?


Theorems
Fully verified type-checker
Normalisation
Inversions
Coq in Coq
AÖV18...
MetaRocq
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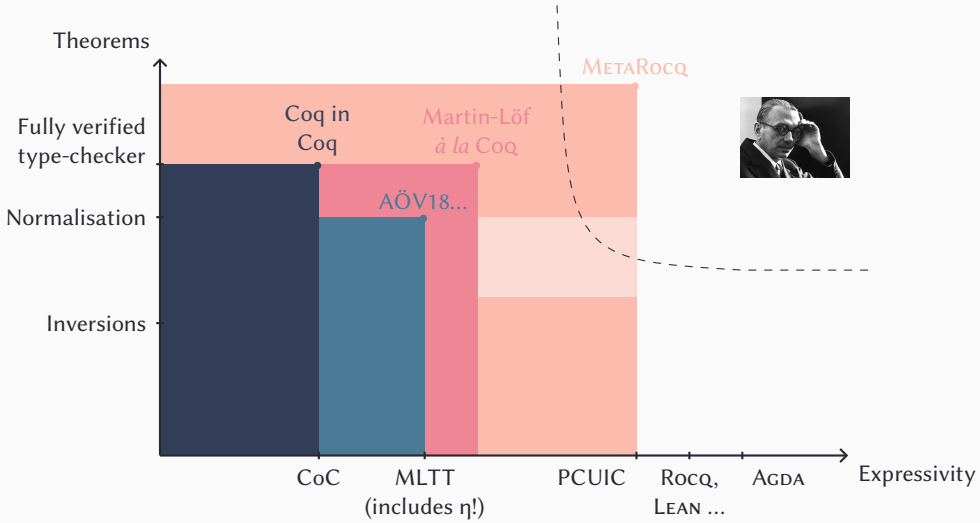
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# Conversion is bidirectional too !

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even if it does not compute types!

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The important part is **invariant preservation**.

**Positive soundness**

Requires inversion for types.

**Negative soundness**

Requires inversion for types and terms.

In particular: **no normalisation**!
Conjecture: requires **low logical power**.

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**Termination**

Requires inversion for types and normalisation.

# How to prove the properties?

| | Logical relation [AÖV17; Adj+24] | Gluing/Nf Model [Ste21; BKS23] | Rewriting/Confluence [Tak95]/MetaRocq | Domain model [CH18] |
|---|---|---|---|---|
| Weak ambiant theory | × | × | ✓ | ✓ |
| Normalisation | ✓ | ✓ | × | × |
| η laws | ✓ | ✓ | × | ✓ |
| Formalised | ✓ | ✓ / × | ✓ | × |
| | Most explored<br>Difficult to scale | Formalisation currently difficult | Insane scaling | Very unexplored |

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**There is space for exploration!**

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**There is space for exploration!**

... but I have peculiar requirements.

Thank you!

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