Type theory as a topic of mathematical study

An introduction to Generalised Algebraic Theories

Formal proof and synthetic mathematics workshop – June 26th 2026

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A bit of context

We've been doing quite a bit of type theory...
Or, to be precise, worked *in* type theories.

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- What is a model of such a type theory?
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Or, to be precise, worked *in* type theories.

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- What is a model of such a type theory?
- What does it mean "to work in" such a type theory?

These questions can be given mathematical answers

Warmup: Universal algebra

Who here has been exposed to universal algebra?

Signature

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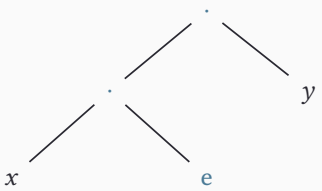
- a set of operations (generators) op_Σ, specified by an arity $\mathrm{ar}_\Sigma \in \mathrm{op}_\Sigma \to \mathbb{N}$

Signature of monoids :

- $\mathrm{op} := \{e, \cdot\}, \quad \mathrm{ar}(e) := 0, \quad \mathrm{ar}(\cdot) := 2$

Syntax trees



- They exist!
- Nice representation in (functional) programming languages (algebraic datatypes)

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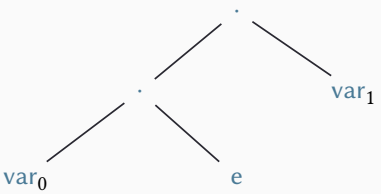

- They exist!
- Nice representation in (functional) programming languages (algebraic datatypes)
- Variables complicate things, but there are ways

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  $\longrightarrow$ generates a set of terms $\mathsf{Tm} \in \mathbb{N} \to \mathbf{Set}$

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  $\longrightarrow$ generates a set of terms $\mathsf{Tm} \in \mathbb{N} \to \mathbf{Set}$
- for each n, a set of equations (relations) $\mathrm{eqs}_\Sigma(n) \subseteq \mathsf{Tm}(n) \times \mathsf{Tm}(n)$

Signature of monoids :

- $\mathrm{op} := \{\mathsf{e}, \cdot\}, \quad \mathrm{ar}(\mathsf{e}) := 0, \quad \mathrm{ar}(\cdot) := 2$
- $\mathsf{e} \in \mathsf{Tm}(0), \quad (x \cdot \mathsf{e}) \cdot y \in \mathsf{Tm}(2)$
- $\mathrm{eqs}(1) := \{(x \cdot \mathsf{e} = x), (\mathsf{e} \cdot x = x)\}, \quad \mathrm{eqs}(3) := \{x \cdot (y \cdot z) = (x \cdot y) \cdot z\}$

Models

Model of a signature (sort, op, ar)

- a set X

A monoid = a model of the signature of monoids:

- a set X

Models

Model of a signature (sort, op, ar)

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- for every operation $o \in \mathrm{op}$ with a function $f_o \in X^{\mathrm{ar}(o)} \to X$

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- a constant $e \in X$ and a function $f \in X \times X \to X$

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- for every operation $\mathrm{o} \in \mathrm{op}$ with a function $f_{\mathrm{o}} \in X^{\mathrm{ar}(\mathrm{o})} \to X$
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- $[\![\mathrm{e}]\!] = e, \quad [\![(x \cdot \mathrm{e}) \cdot y]\!](u, v) = f(f(u, e), v)$

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- such that for every $(l, r) \in \mathrm{eqs}(n)$ and $v \in X^n$, we have $[\![l]\!](v) = [\![r]\!](v)$

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- such that for all u, v and w we have $f(e, v) = v$, $f(v, e) = v$ and $f(u, f(v, w)) = f(f(u, v), w)$

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+ **Morphism of model**

Free/initial model

Theorem (originally, Birkhoff?)

Every signature has a **free model**, which is initial amongst all models.

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Free model for the signature of monoids	=	free monoid = $\{e\}$
Free model for the signature of monoids + 1 constant per $a \in A$	=	free monoid on A

Generalised algebraic theory

Multi-sorted universal algebra

Monoids, rings, propositional logic… fit in universal algebra
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- a set of operations op_Σ, specified by an arity $\mathrm{ar}_\Sigma \in \mathrm{op}_\Sigma \to {\mathrm{sort}_\Sigma}^\star \times \mathrm{sort}_\Sigma$
  $\longrightarrow$ generates a set of terms $\mathrm{Tm} \in {\mathrm{sort}_\Sigma}^\star \times \mathrm{sort}_\Sigma \to \mathbf{Set}$
- a set of equations $\mathrm{eqs}_\Sigma(v, s) \subseteq \mathrm{Tm}(v, s) \times \mathrm{Tm}(v, s)$

Multi-sorted universal algebra

Monoids, rings, propositional logic... fit in universal algebra
Modules, vector spaces... need multiple sorts

- two sorts = s for scalars and v for vectors
- operations $\mathrm{ar}(\cdot_v) := ([s, v], v)$ $\mathrm{ar}(\cdot_s) := ([s, s], s)$ $\mathrm{ar}(+_s) := ([s, s], s)$ $\mathrm{ar}(+_v) := ([v, v], v)$
- + equations

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Still not enough!

But there's more!

- Category: set of morphisms $\mathrm{Hom}(X, Y)$ for objects X, Y
- First-order logic: expressions, formulas over a given context
- Type theory: types over a context, terms of a type in a context

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- Category: set of morphisms $\mathrm{Hom}(X, Y)$ for objects X, Y
- First-order logic: expressions, formulas over a given context
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Missing ingredient = dependent sorts

More precisely

A category:

- sorts $\mathrm{ob} \in \mathbf{Set}$ and $\mathrm{hom} \in \prod_{x,y \in \mathrm{ob}} \mathbf{Set}$
- operations $\mathrm{id} \in \prod_{x \in \mathrm{ob}} \mathrm{hom}(x,x)$ and $\circ \in \prod_{g \in \mathrm{hom}(y,z), f \in \mathrm{hom}(x,y)} \mathrm{hom}(x,z)$
- equations $\prod_{f \in \mathrm{hom}(x,y)} f \circ \mathrm{id}(x) = f$ etc.

More precisely

A category:

- sorts $\mathsf{ob} \in \mathbf{Set}$ and $\mathsf{hom} \in \prod_{x,y \in \mathsf{ob}} \mathbf{Set}$
- operations $\mathsf{id} \in \prod_{x \in \mathsf{ob}} \mathsf{hom}(x,x)$ and $\circ \in \prod_{g \in \mathsf{hom}(y,z), f \in \mathsf{hom}(x,y)} \mathsf{hom}(x,z)$
- equations $\prod_{f \in \mathsf{hom}(x,y)} f \circ \mathsf{id}(x) = f$ etc.

A (model of) dependent type theory:

- $\mathsf{Ctx} \in \mathbf{Set}$ (contexts)
- $\mathsf{Ty} \in \mathsf{Ctx} \to \mathbf{Set}$ (types)
- $\mathsf{Tm} \in \prod_{\Gamma \in \mathsf{Ctx}} \mathsf{Ty}(\Gamma) \to \mathbf{Set}$ (terms)
- ...

Use a type theory (= the logical framework) to define signatures

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Signature, take 2

A signature is a context in the type theory below.

$$\frac{}{\Sigma \vdash \mathsf{Sort}} \qquad \frac{\Sigma \vdash X : \mathsf{Sort}}{\Sigma \vdash \mathsf{El}(X)}$$

$$\frac{\Sigma \vdash A : \mathsf{Sort} \qquad \Sigma \vdash l : \mathsf{El}(A) \qquad \Sigma \vdash r : \mathsf{El}(A)}{\Sigma \vdash l \equiv_A r} \qquad \text{(extensional)}$$

$$\Sigma, (X\colon \mathsf{Sort}) \qquad \Sigma', (x\colon \mathsf{El}(X)) \qquad \Sigma'', (e\colon x \equiv_A y)$$

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$$\frac{\Sigma \vdash A : \mathsf{Sort} \qquad \Sigma, (x{:}\,A) \vdash B}{\vdash (x{:}\,A) \to B} \qquad \frac{\Sigma \vdash f : (x{:}\,A) \to B \qquad \Sigma \vdash u : \mathsf{El}(A)}{\Sigma \vdash f\ u : \mathsf{El}(B[u])}$$

$$\Sigma, (X : \mathsf{Sort}) \qquad \Sigma', (f : \mathsf{El}(X) \to \mathsf{El}(X)) \qquad \Sigma'', (e : (x{:}\,\mathsf{El}(X)) \to f\ (f\ x) \equiv_X x)$$

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For now: work with this type theory informally

The GAT of categories, very formally

$$
\begin{aligned}
\mathsf{ob} &: \mathsf{Sort} \\
\mathsf{hom} &: (x : \mathsf{ob}) \to (y : \mathsf{ob}) \to \mathsf{Sort} \\
\\
\mathsf{id} &: (x : \mathsf{ob}) \to \mathsf{El}(\mathsf{hom}(x, x)) \\
\mathsf{comp} &: (x, y, z : \mathsf{ob}) \to (f : \mathsf{hom}(x, y)) \to (g : \mathsf{hom}(y, z)) \to \mathsf{El}(\mathsf{hom}(x, z)) \\
\\
\mathsf{idl} &: (x, y : \mathsf{ob}) \to (f : \mathsf{hom}(x, y)) \to \mathsf{comp}(x, x, y, \mathsf{id}(x), f) \equiv_{\mathsf{hom}(x,y)} f \\
\mathsf{idr} &: (x, y : \mathsf{ob}) \to (f : \mathsf{hom}(x, y)) \to \mathsf{comp}(x, y, y, f, \mathsf{id}(y)) \equiv_{\mathsf{hom}(x,y)} f \\
\mathsf{assoc} &: (w, x, y, z : \mathsf{ob}) \to (f : \mathsf{hom}(w, x)) \to (g : \mathsf{hom}(x, y)) \to (h : \mathsf{hom}(y, z)) \to \\
&\qquad \mathsf{comp}(w, x, z, f, \mathsf{comp}(x, y, z, g, h)) \equiv_{\mathsf{hom}(w,z)} \mathsf{comp}(w, y, z\, \mathsf{comp}(w, x, y, f, g), h)
\end{aligned}
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&\qquad \mathsf{comp}(w, x, z, f, \mathsf{comp}(x, y, z, g, h)) \equiv_{\mathsf{hom}(w,z)} \mathsf{comp}(w, y, z\, \mathsf{comp}(w, x, y, f, g), h)
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What a mouthful!

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\\
\mathsf{id} &: \{x : \mathsf{ob}\} \to \mathsf{El}(\mathsf{hom}(x, x)) \\
\circ &: \{x, y, z : \mathsf{ob}\} \to (g : \mathsf{hom}(y, z)) \to (f : \mathsf{hom}(x, y)) \to \mathsf{El}(\mathsf{hom}(x, z)) \\
\\
& (x, y : \mathsf{ob}) \to (f : \mathsf{hom}(x, y)) \to f \circ \mathsf{id}_x \equiv f \\
& (x, y : \mathsf{ob}) \to (f : \mathsf{hom}(x, y)) \to \mathsf{id}_y \circ f \equiv f \\
& (w, x, y, z : \mathsf{ob}) \to (f : \mathsf{hom}(w, x)) \to (g : \mathsf{hom}(x, y)) \to (h : \mathsf{hom}(y, z)) \to \\
& \qquad (h \circ g) \circ f \equiv h \circ (g \circ f)
\end{aligned}
$$

What a mouthful! Conventions: implicit arguments + notations

The GAT of categories, very formally

$$\frac{}{\vdash \mathrm{ob}} \qquad \frac{\vdash x : \mathrm{ob} \qquad \vdash y : \mathrm{ob}}{\vdash \mathrm{hom}(x, y)}$$

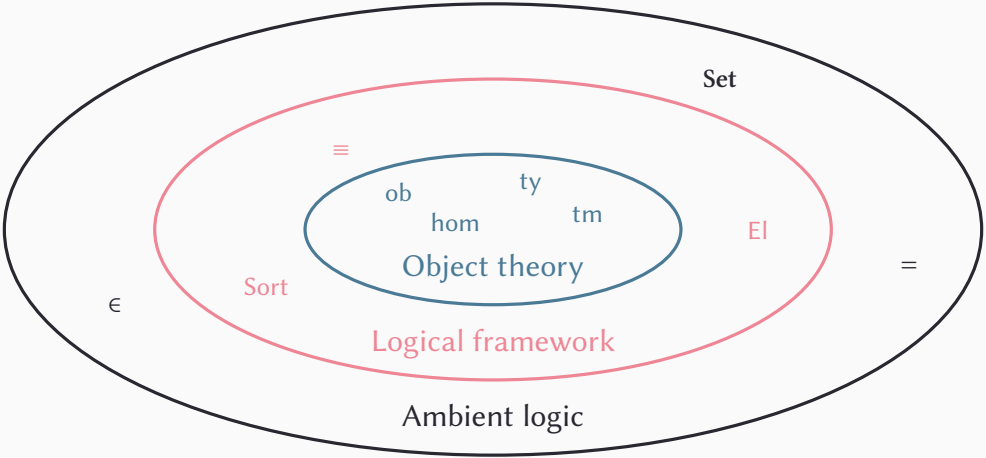
$$\frac{\vdash x : \mathrm{ob}}{\vdash \mathrm{id}_x : \mathrm{hom}(x, y)} \qquad \frac{\vdash x, y, z : \mathrm{ob} \qquad \vdash f : \mathrm{hom}(x, y) \qquad \vdash g : \mathrm{hom}(y, z)}{\vdash g \circ f : \mathrm{hom}(x, z)}$$

$$\frac{\vdash x, y : \mathrm{ob} \qquad \vdash f : \mathrm{hom}(x, y)}{\vdash f \circ \mathrm{id}_x \equiv f : \mathrm{hom}(x, y)} \qquad \frac{\vdash x, y : \mathrm{ob} \qquad \vdash f : \mathrm{hom}(x, y)}{\vdash \mathrm{id}_y \circ f \equiv f : \mathrm{hom}(x, y)}$$

$$\frac{\vdash w, x, y, z : \mathrm{ob} \qquad \vdash f : \mathrm{hom}(w, x) \qquad \vdash g : \mathrm{hom}(x, y) \qquad \vdash h : \mathrm{hom}(y, z)}{\vdash (h \circ g) \circ f \equiv h \circ (g \circ f) : \mathrm{hom}(w, z)}$$

What a mouthful! As typing rules

The type theorist's favourite GAT: Categories with Families



The type theorist's favourite GAT: Categories with Families

$$\begin{aligned} \mathsf{ctx} &: \mathsf{Sort} \\ \mathsf{sub} &: (\Gamma : \mathsf{ctx}) \to (\Delta : \mathsf{ctx}) \to \mathsf{Sort} \\ &\dots \end{aligned}$$

ctx and sub form a category

The type theorist's favourite GAT: Categories with Families

$$
\begin{aligned}
&\mathsf{ctx} : \mathsf{Sort} \\
&\mathsf{sub} : (\Gamma : \mathsf{ctx}) \to (\Delta : \mathsf{ctx}) \to \mathsf{Sort} \\
&\quad \dots \\
&\mathsf{ty} : (\Gamma : \mathsf{ctx}) \to \mathsf{Sort} \\
&\_[\_] : \{\Gamma, \Delta : \mathsf{ctx}\} \to (X : \mathsf{ty}(\Delta)) \to (\sigma : \mathsf{sub}(\Gamma, \Delta)) \to \mathsf{El}(\mathsf{ty}(\Gamma)) \\
&\quad (\Gamma : \mathsf{ctx}) \to (X : \mathsf{ty}(\Gamma)) \to X[\mathsf{id}] \equiv X \\
&\quad \dots
\end{aligned}
$$

ctx and sub form a category

ty is a functor

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$$
\begin{array}{rll}
\mathsf{ctx} : & \mathsf{Sort} & \\
\mathsf{sub} : & (\Gamma : \mathsf{ctx}) \to (\Delta : \mathsf{ctx}) \to \mathsf{Sort} & \mathsf{ctx} \text{ and } \mathsf{sub} \text{ form a category} \\
& \dots & \\
 & & \\
\mathsf{ty} : & (\Gamma : \mathsf{ctx}) \to \mathsf{Sort} & \\
\_[\_] : & \{\Gamma, \Delta : \mathsf{ctx}\} \to (X : \mathsf{ty}(\Delta)) \to (\sigma : \mathsf{sub}(\Gamma, \Delta)) \to \mathsf{El}(\mathsf{ty}(\Gamma)) & \\
 & (\Gamma : \mathsf{ctx}) \to (X : \mathsf{ty}(\Gamma)) \to X[\mathsf{id}] \equiv X & \mathsf{ty} \text{ is a functor} \\
 & \dots & \\
 & & \\
\mathsf{tm} : & (\Gamma : \mathsf{ctx}) \to (X : \mathsf{ty}(\Gamma)) \to \mathsf{Sort} & \\
\_[\_] : & \{\Gamma, \Delta : \mathsf{ctx}\} \to \{X : \mathsf{ty}(\Delta)\} \to (x : \mathsf{tm}(\Delta, X)) \to (\sigma : \mathsf{sub}(\Gamma, \Delta)) \to & \\
 & \quad \mathsf{El}(\mathsf{tm}(\Gamma, X[\sigma])) & \mathsf{tm} \text{ is a displayed functor} \\
 & (\Gamma : \mathsf{ctx}) \to (X : \mathsf{ty}(\Gamma)) \to (x : \mathsf{tm}(\Gamma, X)) \to x[\mathsf{id}] \equiv_{\mathsf{tm}(\Gamma, X)} x & \\
 & \dots &
\end{array}
$$

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$$\begin{aligned}
&\mathsf{ctx} : \mathsf{Sort} \\
&\mathsf{sub} : (\Gamma : \mathsf{ctx}) \to (\Delta : \mathsf{ctx}) \to \mathsf{Sort} && \text{ctx and sub form a category} \\
&\dots \\
&\mathsf{ty} : (\Gamma : \mathsf{ctx}) \to \mathsf{Sort} \\
&_[_] : \{\Gamma, \Delta : \mathsf{ctx}\} \to (X : \mathsf{ty}(\Delta)) \to (\sigma : \mathsf{sub}(\Gamma, \Delta)) \to \mathsf{El}(\mathsf{ty}(\Gamma)) \\
&\quad (\Gamma : \mathsf{ctx}) \to (X : \mathsf{ty}(\Gamma)) \to X[\mathsf{id}] \equiv X && \text{ty is a functor} \\
&\quad \dots \\
&\mathsf{tm} : (\Gamma : \mathsf{ctx}) \to (X : \mathsf{ty}(\Gamma)) \to \mathsf{Sort} \\
&_[_] : \{\Gamma, \Delta : \mathsf{c} \\
&\quad \mathsf{El}(\mathsf{tr} && \text{nctor} \\
&\quad (\Gamma : \mathsf{ctx}) \\
&\quad \dots
\end{aligned}$$

$$(\Gamma : \mathsf{ctx}) \to (X : \mathsf{ty}(\Gamma)) \to (x : \mathsf{tm}(\Gamma, X)) \to x[\mathsf{id}] \equiv_{\mathsf{tm}(\Gamma, X)} x$$

The type theorist's favourite GAT: Categories with Families (Ctd.)

$\diamond :\quad \mathsf{El}(\mathsf{ctx})$

$! :\quad \{\Gamma : \mathsf{ctx}\} \to \mathsf{El}(\mathsf{sub}(\Gamma, \diamond))$

$(\Gamma : \mathsf{ctx}) \to (\sigma : \mathsf{sub}(\Gamma, \diamond)) \to \sigma \equiv !$

ctx has a terminal object (empty context)

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$\diamond :\quad \mathsf{El}(\mathsf{ctx})$

$! :\quad \{\Gamma : \mathsf{ctx}\} \to \mathsf{El}(\mathsf{sub}(\Gamma, \diamond))$ — ctx has a terminal object (empty context)

$\quad\quad (\Gamma : \mathsf{ctx}) \to (\sigma : \mathsf{sub}(\Gamma, \diamond)) \to \sigma \equiv !$

$\_ \triangleright \_ :\quad (\Gamma : \mathsf{ctx}) \to (X : \mathsf{ty}(\Gamma)) \to \mathsf{El}(\mathsf{ctx})$ — context extension $(\Gamma, x : X)$

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$$
\begin{array}{rll}
\diamond : & \mathsf{El}(\mathsf{ctx}) & \\
! : & \{\Gamma : \mathsf{ctx}\} \to \mathsf{El}(\mathsf{sub}(\Gamma, \diamond)) & \text{ctx has a terminal object (empty context)} \\
 & (\Gamma : \mathsf{ctx}) \to (\sigma : \mathsf{sub}(\Gamma, \diamond)) \to \sigma \equiv ! & \\
 & & \\
\_ \triangleright \_ : & (\Gamma : \mathsf{ctx}) \to (X : \mathsf{ty}(\Gamma)) \to \mathsf{El}(\mathsf{ctx}) & \text{context extension } (\Gamma, x : X) \\
\mathsf{var} : & \{\Gamma : \mathsf{ctx}\} \to \{X : \mathsf{ty}(\Gamma)\} \to \mathsf{El}(\mathsf{tm}(\Gamma \triangleright X, X\quad)) & \text{last variable in context}
\end{array}
$$

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$\diamond :$	$\mathsf{El}(\mathsf{ctx})$	
$! :$	$\{\Gamma : \mathsf{ctx}\} \to \mathsf{El}(\mathsf{sub}(\Gamma, \diamond))$	ctx has a terminal object (empty context)
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$\_ \triangleright \_ :$	$(\Gamma : \mathsf{ctx}) \to (X : \mathsf{ty}(\Gamma)) \to \mathsf{El}(\mathsf{ctx})$	context extension $(\Gamma, x : X)$
$\mathsf{var} :$	$\{\Gamma : \mathsf{ctx}\} \to \{X : \mathsf{ty}(\Gamma)\} \to \mathsf{El}(\mathsf{tm}(\Gamma \triangleright X, X[\uparrow]))$	last variable in context
$\uparrow :$	$\{\Gamma : \mathsf{ctx}\} \to \{X : \mathsf{ty}(\Gamma)\} \to \mathsf{El}(\mathsf{sub}(\Gamma \triangleright X, \Gamma))$	weakening

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! : & \{\Gamma : \mathsf{ctx}\} \to \mathsf{El}(\mathsf{sub}(\Gamma, \diamond)) & \mathsf{ctx} \text{ has a terminal object (empty context)} \\
 & (\Gamma : \mathsf{ctx}) \to (\sigma : \mathsf{sub}(\Gamma, \diamond)) \to \sigma \equiv ! & \\
 & & \\
\_ \triangleright \_ : & (\Gamma : \mathsf{ctx}) \to (X : \mathsf{ty}(\Gamma)) \to \mathsf{El}(\mathsf{ctx}) & \text{context extension } (\Gamma, x : X) \\
\mathsf{var} : & \{\Gamma : \mathsf{ctx}\} \to \{X : \mathsf{ty}(\Gamma)\} \to \mathsf{El}(\mathsf{tm}(\Gamma \triangleright X, X[\uparrow])) & \text{last variable in context} \\
\uparrow : & \{\Gamma : \mathsf{ctx}\} \to \{X : \mathsf{ty}(\Gamma)\} \to \mathsf{El}(\mathsf{sub}(\Gamma \triangleright X, \Gamma)) & \text{weakening} \\
(\_, , \_) : & \{\Gamma, \Delta : \mathsf{ctx}\} \to \{X : \mathsf{ty}(\Gamma)\} \to (\sigma : \mathsf{sub}(\Delta, \Gamma)) \to & \text{substitution extension} \\
 & \quad (x : \mathsf{tm}(\Delta, X[\sigma])) \to \mathsf{El}(\mathsf{sub}(\Delta, \Gamma \triangleright X)) &
\end{array}
$$

+ equations

The type theorist's favourite GAT: Categories with Families (Ctd.)

$$
\begin{array}{rll}
\diamond : & \mathrm{El}(\mathsf{ctx}) & \\
! : & \{\Gamma : \mathsf{ctx}\} \to \mathrm{El}(\mathsf{sub}(\Gamma, \diamond)) & \mathsf{ctx} \text{ has a terminal object (empty context)} \\
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+ equations

Only terms: $\mathsf{var}[\uparrow^n]$ = n-th variable in context

Models of a GAT

The logical framework looks a lot like the ambient logic...

The standard model

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(\_) \to \_ & \mapsto & \prod \\
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Definition

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In fact, we have a **category** $\mathrm{Mod}(\Sigma)$ (might be too strict to your liking!)

What is a CwF?

Read from the GAT definition! Or...

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Definition (Fiore 2012, Awodey 2018)

A **natural model** consists of:

- a category $\mathscr{C}$ with a terminal object = contexts + substitutions + empty context
- two presheaves Ty and Tm (functors $\mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$) = types, terms, action of substitutions
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"$\Gamma \triangleright A$ is the universal context which extends Γ and has a term of type A"

$$\begin{array}{ccc} \mathrm{y}(\Gamma \triangleright A) & \xrightarrow{\mathrm{var}} & Tm \\ \uparrow \downarrow & \lrcorner & \downarrow ty \\ \mathrm{y}\,\Gamma & \xrightarrow[A]{} & Ty \end{array}$$

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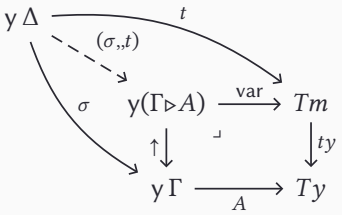

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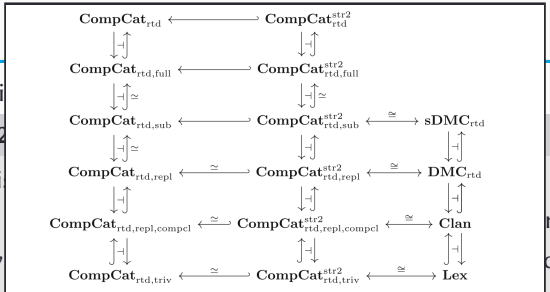

Fig. 4. Notions with types as maps, rooted

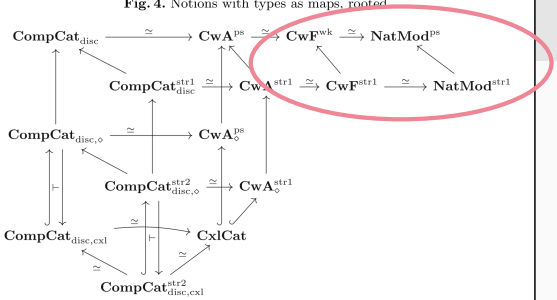

Fig. 5. Notions with types primitive

Syntax

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Constructions that are valid in every model?

Syntax is the initial model

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Some sort of manipulation of syntax trees?

It is the same!

Why are syntax tree not enough?

Ill-formed syntax trees: $0 + +$ $\lambda\, x.y$

Ill-typed syntax trees: $0 + (\lambda\, x.x)$

$\longrightarrow$ some syntax trees do not represent any valid terms

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Equations: $\mathrm{id} \circ f = f$
$\longrightarrow$ many syntax trees represent equal terms

If f represents a valid term of type $\hom(a, b)$, is $f \circ \mathrm{id}_{a'}$ valid?
$\longrightarrow$ equations play a role in typing!

Theorem (Cartmell 1986, Kaposi, Kovács and Altenkirch 2019)

The initial model of any GAT exists.

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Implementation: work with pre-terms, but check for typing/definitional equality

Much better if typing is **decidable**!

Wrapping up

One type theory to describe them all: a **logical framework**

Do the work **once**, get it for **all theories**:

- syntax
- (category of) models

Syntax is the **initial model**

Where to go from here?

To go on with GATs and CwFs

- *Generalised algebraic theories and contextual categories* (Cartmell 1986)
- *Categories with families: Unityped, simply typed, and dependently typed* (Castellan et al. 2021)
- *Constructing Quotient Inductive-Inductive Types* (Kaposi et al. 2019)
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Other logical frameworks

- SOGATs = GATs + built-in handling of contexts & variables
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- a more syntactic version (// construction of syntax)
  → *A general definition of dependent type theories* (Bauer et al. 2020)
- proof assistants!
  → Isabelle, Dedukti, Andromeda 2, Beluga...

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Theorem (Uemura 2022)

The category of GATs is the bi-initial category with finite limits and an exponentiable morphism.

Thank you!

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