Bidirectional Interpolation for the λ-Calculus

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Meven Lennon-Bertrand Alexis Saurin

INRIA – IRIF, Université Paris Cité

If $F \vdash G$ then there exists M such that

- $F \vdash M \vdash G$
- $\mathrm{voc}(M) \subseteq \mathrm{voc}(F) \cap \mathrm{voc}(G)$

Interpolation

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A **factorisation** property of sorts (ask Noam for the categorical view)... which logicians **love**!

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Alexis (a logician): “It would be cool to prove an interpolation theorem for dependent types!”
Me (a type theorist): “Sure, how does an interpolation proof look like?”

Simply-typed λ-calculus ($\times, \top, \to, +, \bot$) has the interpolation property (Čubrić, 1994)

If $x : A \vdash t : B$ then there exist M, r, l such that:

- $x : A \vdash l : M$
- $y : M \vdash r : B$
- $r[l] \equiv t$
- $\mathrm{voc}(M) \subseteq \mathrm{voc}(A) \cap \mathrm{voc}(B)$ (base types appearing in M appear both in A and B)

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If $x\colon A \vdash t : B$ then there exist M, r, l such that:

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Internal language of biCCCs $\longrightarrow$ nice consequences for the bicategory of biCCCs

Proof-relevant interpolation

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Great!

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Internal language of biCCCs $\longrightarrow$ nice consequences for the bicategory of biCCCs

Great! Or?

Simply- ... **994)**

If $x\colon A \vdash$ ...

- $x\colon A$ ...
- $y\colon M$ ...
- $r[l] \equiv$ ...
- voc(...

Internal la...

Great! Or?

First we need the following lemma (cf. [Pra65, Cor. 3, p. 54]).

Lemma 3.5. *For a free $\lambda\delta$-calculus (no additional equations but with constants) the following holds: let t^C be a ρ-normal term which is one of: $\pi_i(s)$, $s`r$, $\delta x\cdot u, y\cdot v; s$, $\varepsilon^C(s)$ or an atomic term (i.e. a variable or a constant (or $*$)). Then there exists a chain $t \equiv t_0 \succ \ldots \succ t_n$ of successive subterms of t such that they are in the following relation: for every $0 \leqq i < n$, t_i is one of $\pi_j(t_{i+1})$, $t_{i+1}`u$, $\delta x\cdot u, y\cdot v; t_{i+1}$ (for some u, v, x and y) or $t_i \equiv \varepsilon^B(t_{i+1})$, and moreover t_n is an atomic term (i.e. $t_n \equiv v^E$ where v is a variable or a constant (or $*$)).*

In particular if the term t is not $*$ then t_n is not $*$. So, if in addition the $\lambda\delta$-calculus doesn't have constants then t_n is a variable.

Furthermore, by C-normality we have: if $t_{n-1} \equiv \delta x\cdot u, y\cdot v; t_n$ then the above chain $t \equiv t_0 \succ \ldots \succ t_n$ is actually just $t \equiv \delta x\cdot u, y\cdot v; t_1$ and t_1 is either $x^{E_1+E_2}$ or a constant (for some ρ-normal terms u, v).

Similarly, by E-normality, if $t_{n-1} \equiv \varepsilon^B(t_n)$ then $t \equiv \varepsilon^B(t_1)$ and t_1 is a variable x^0 or a constant of type 0.

Simply- … 994)

If $x: A \vdash$

- $x: A$
- $y: M$
- $r[l] \equiv$
- voc(

Internal la

Great! Or?

Case 6: (In this case only, we use Lemma 3.5.) $t\equiv\pi_i(s)$ or $t\equiv s`r$ or $t\equiv\delta x\cdot u, y\cdot v; s$ or $t\equiv\varepsilon^C(s)$ where r, s, u and v are some ρ-normal terms. Then by Lemma 3.5 we have that there exists a chain of immediate subterms: $t\equiv t_0\succ t_1\succ\ldots\succ t_i\succ\ldots\succ t_n$ such that the following subcases can take place: $t_{n-1}^{E_i}\equiv\pi_i(x^{E_1\times E_2})$ or $t_{n-1}^{E_1}\equiv x^{E_1^{E_2}}`u^{E_2}$ or $t_{n-1}\equiv\delta x\cdot u, y\cdot v; x^{E_1+E_2}$ or $t_{n-1}\equiv\varepsilon^B(x^0)$ for some ρ-normal terms u and v since in the calculus we don't have additional constants and $t\not\equiv *$.

Subcase 6.1.1: $t_{n-1}^{E_i}\equiv\pi_i(x^{E_1\times E_2})$ and $x^{E_1\times E_2}\in\Gamma_1$.

Let t' be like the term t expect that it has x^{E_i} instead of t_{n-1} (notice however, that t' can contain $x^{E_1\times E_2}$) so $t=_\Gamma t'(\pi_i(x^{E_1\times E_2})/x^{E_i})$. The complexity of t' is lower then the complexity of t, so we can apply the induction hypothesis on $(\Gamma\cup\{x^{E_i}\}\rhd t')$ and the partition $\Gamma, x^{E_i}=(\Gamma_1, x^{E_i})\cup\Gamma_2$. Then by the induction hypothesis then exist $(\Gamma_1, x^{E_i}\rhd R^A)$ and $(\Gamma_2, y^A\rhd S^C)$ such that

$$t'=_{\Gamma, x^{E_i}}S(R/y),\qquad A^+\subseteq(\Gamma_1\cup E_i)^+\cap(\Gamma_2^-\cup C^+),\qquad A^-\subseteq(\Gamma_1\cup E_i)^-\cap(\Gamma_2^+\cup C^-).$$

Recall that $x^{E_1\times E_2}$ is in Γ_1. So we define $r=R(\pi_i(x^{E_1\times E_2}))$, $s=S$ and A stays the same and we can check that the lemma is satisfied. First:

$$t=_\Gamma t'(\pi_i(x^{E_1\times E_2})/x^{E_i})=_\Gamma S((R(\pi_i(x^{E_1\times E_2})/x^{E_i}))/y)\equiv s(r/y).$$

The second and third part of the conclusion are satisfied since $E_i^s\subset(E_1\times E_2)^s$ where $s=+,-$. It is obvious that the stronger hypothesis gives the reduction instead of the equality.

Simply- ... **(1994)**

If $x{:}\,A \vdash$

- $x{:}\,A$
- $y{:}\,M$
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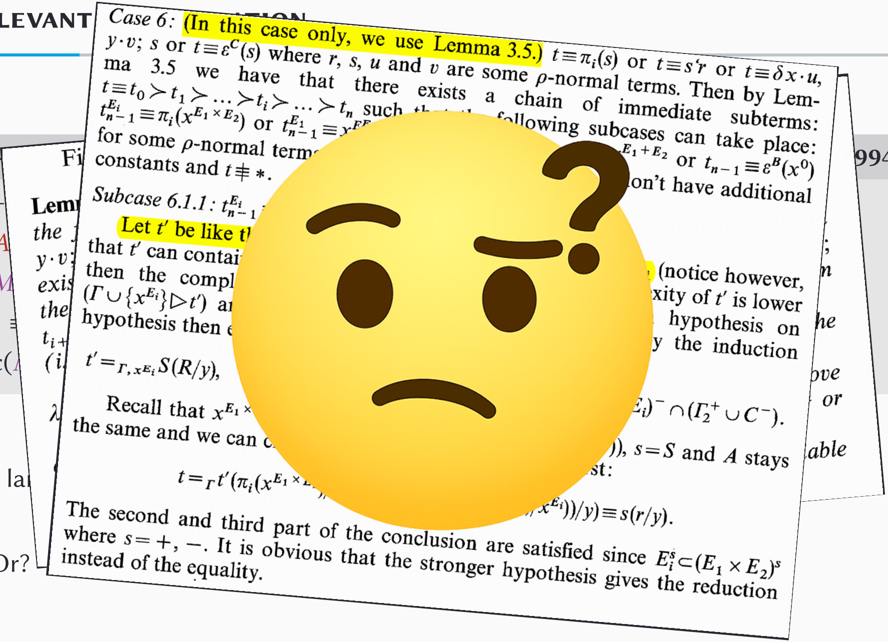
Case 6: (In this case only, we use Lemma 3.5.) $t \equiv \pi_i(s)$ or $t \equiv s`r$ or $t \equiv \delta x \cdot u$, $y \cdot v$; s or $t \equiv \varepsilon^C(s)$ where r, s, u and v are some ρ-normal terms. Then by Lemma 3.5 we have that there exists a chain of immediate subterms: $t \equiv t_0 \succ t_1 \succ \ldots \succ t_i \succ \ldots \succ t_n$ such th... following subcases can take place: $t_{n-1}^{E_i} \equiv \pi_i(x^{E_1 \times E_2})$ or $t_{n-1}^{E_1} \equiv x^{F...}$ ... ${}^{E_1+E_2}$ or $t_{n-1} \equiv \varepsilon^B(x^0)$ for some ρ-normal term... ...on't have additional constants and $t \not\equiv *$.

Subcase 6.1.1: $t_{n-1}^{E_i}$ ...

Let t' be like t... that t' can contai... (notice however, then the compl... ...xity of t' is lower $(\Gamma \cup \{x^{E_i}\} \rhd t')$ a... hypothesis on hypothesis then e... ...y the induction

$t' =_{\Gamma, x^{E_i}} S(R/y)$,

Recall that $x^{E_1 \times}$ ... $E_i)^- \cap (\Gamma_2^+ \cup C^-)$. the same and we can c... ...)), $s = S$ and A stays ...st:

$t =_\Gamma t'(\pi_i(x^{E_1 \times E}$ ... $x^{E_i}))/y) \equiv s(r/y)$.

The second and third part of the conclusion are satisfied since $E_i^s \subset (E_1 \times E_2)^s$ where $s = +, -$. It is obvious that the stronger hypothesis gives the reduction instead of the equality.

Internal la

Great! Or?

That's just neutral terms!

$$\frac{}{\mathsf{nf}\ *} \qquad \frac{\mathsf{nf}\ t \qquad \mathsf{nf}\ u}{\mathsf{nf}\ (t, u)} \qquad \frac{\mathsf{nf}\ t}{\mathsf{nf}\ \lambda\, x.t} \qquad \frac{\mathsf{ne}\ t}{\mathsf{nf}\ t}$$

$$\frac{}{\mathsf{ne}\ x} \qquad \frac{\mathsf{ne}\ p}{\mathsf{ne}\ p_{.i}} \qquad \frac{\mathsf{ne}\ t \qquad \mathsf{nf}\ u}{\mathsf{ne}\ t\ u}$$

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Why neutrals/normals?

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Why neutrals/normals?
Because they represent equivalence classes for $\equiv_\beta$

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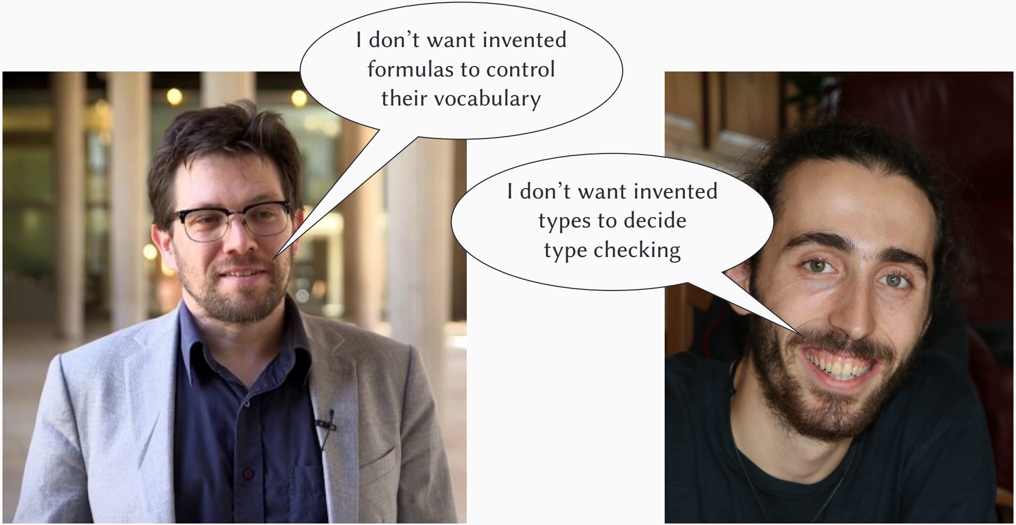
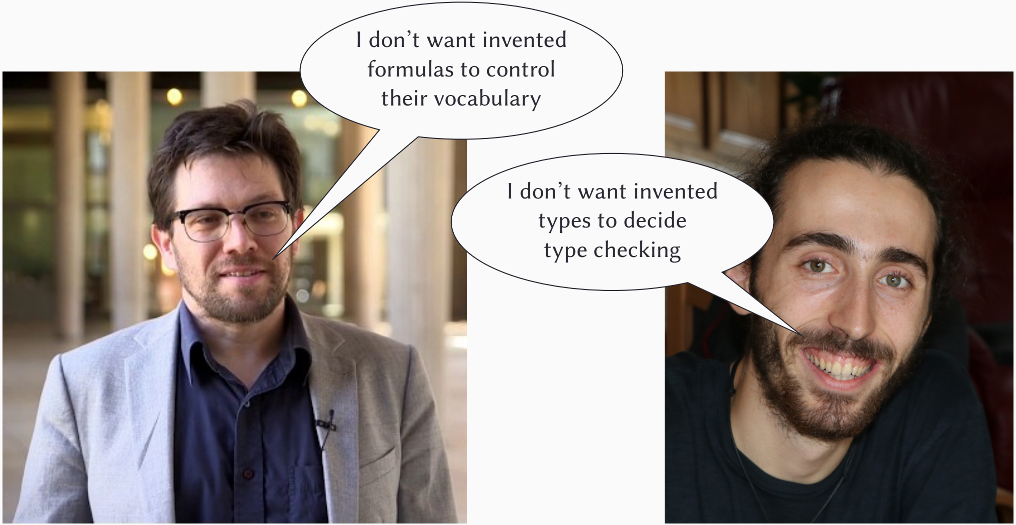
Why neutrals/normals?
~~Because they represent equivalence classes~~
Because they have a good **subformula property** → formulas always come from somewhere


I don’t want invented formulas to control their vocabulary


I don’t want invented formulas to control their vocabulary
I don’t want invented types to decide type checking



It's the same thing!

Normal forms check, neutrals infer

$\boxed{\Gamma \vdash t \lhd A}$ Normal/checking forms $t, u, \cdots \in \mathrm{Nf}(\Gamma, A)$

$$\frac{}{\Gamma \vdash * \lhd \top} \qquad \frac{\Gamma \vdash t \lhd A \qquad \Gamma \vdash u \lhd B}{\Gamma \vdash (t, u) \lhd A \times B} \qquad \frac{\Gamma, x : A \vdash t \lhd B}{\Gamma \vdash \lambda\, x.t \lhd A \to B} \qquad \frac{\Gamma \vdash t \rhd A \qquad A = B}{\Gamma \vdash \underline{t} \lhd B}$$

Normal forms check, neutrals infer

$\boxed{\Gamma \vdash t \rhd A}$ Neutral/infering forms $t, u, \cdots \in \mathrm{Ne}(\Gamma, A)$

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \rhd A} \qquad \frac{\Gamma \vdash p \rhd A_1 \times A_2}{\Gamma \vdash p._i \rhd A_i} \qquad \frac{\Gamma \vdash t \rhd A \to B \qquad \Gamma \vdash u \lhd A}{\Gamma \vdash t\ u \rhd B}$$

Normalisation

For any term $\Gamma \vdash t : A$, there exists t' such that $\Gamma \vdash t' \triangleleft A$ and $t \rightsquigarrow^*_\beta t'$.

Logical relations/reducibility/..., nothing new

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Interpolation

Assume a partitioned context $\Gamma = \Gamma_s \cup \Gamma_t$.
Given any $\Gamma \vdash t \triangleleft T$, there is a type M and terms $\Gamma_s \vdash l : M$ and $\Gamma_t.(x\colon M) \vdash r : T$ such that $r[l] \equiv t$ and $\mathrm{voc}(M) \subseteq \mathrm{voc}(\Gamma_s) \cap (\mathrm{voc}(\Gamma_t) \cup \mathrm{voc}(T))$.

Normalisation

For any term $\Gamma \vdash t : A$, there exists t' such that $\Gamma \vdash t' \triangleleft A$ and $t \rightsquigarrow_\beta^* t'$.

Logical relations/reducibility/..., nothing new

Interpolation

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Mutual induction: given any $\Gamma \vdash t \triangleright T$, either

- $\mathrm{voc}(T) \subseteq \mathrm{voc}(\Gamma_s)$, and we have a type M and terms $\Gamma_t \vdash l : M$ and $\Gamma_s.(x\colon M) \vdash r : T$ such that $r[l] \equiv t$ and $\mathrm{voc}(M) \subseteq \mathrm{voc}(\Gamma_s) \cap \mathrm{voc}(\Gamma_t)$.
- or $\mathrm{voc}(T) \subseteq \mathrm{voc}(\Gamma_t)$, and we have a type M and $\Gamma_s \vdash l : M$ and $\Gamma_t.(x\colon M) \vdash r : T$ such that $r[l] \equiv t$ and $\mathrm{voc}(M) \subseteq \mathrm{voc}(\Gamma_s) \cap \mathrm{voc}(\Gamma_t)$.

And sums?

$$\frac{\Gamma \vdash t : \bot}{\Gamma \vdash \mathsf{raise}\, t : \color{red}{T}} \qquad \frac{\Gamma \vdash s : A + B \qquad \Gamma.(x\colon A) \vdash b_l : \color{red}{T} \qquad \Gamma.(x\colon B) \vdash b_r : \color{red}{T}}{\Gamma \vdash \mathsf{if}(s, b_l, b_r) : \color{red}{T}}$$

And sums?


The elimination rules are very bad. What is catastrophic about them is the parasitic presence of a formula T which has no structural link with the formula which is eliminated.
– Proofs and Types

$b_r : T$

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$$\dots$$

$$\frac{\Gamma \vdash a \triangleleft A}{\Gamma \vdash \mathsf{inl}\, a \triangleleft A + B} \qquad \frac{\Gamma \vdash b \triangleleft B}{\Gamma \vdash \mathsf{inr}\, b \triangleleft A + B}$$

$$\frac{\Gamma \vdash t \triangleright \bot}{\Gamma \vdash \mathsf{raise}\, t \triangleleft A} \qquad \frac{\Gamma \vdash s \triangleright A + B \quad \Gamma.(x\colon A) \vdash b_l \triangleleft T \quad \Gamma.(x\colon B) \vdash b_r \triangleleft T}{\Gamma \vdash \mathsf{if}(s, b_l, b_r) \triangleleft T}$$

Good subformula property, although not canonical representatives

$$\cdots$$

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Good subformula property, although not canonical representatives

Normalisation up to β and **commuting conversion**

$$\begin{aligned} (\mathsf{raise}\, t)\ u &\rightsquigarrow^* \mathsf{raise}\, t \\ \mathsf{if}(t, b_l, b_r)\ u &\rightsquigarrow^* \mathsf{if}(t, b_l\ u, b_r\ u) \end{aligned}$$

Normalization by Evaluation for Call-By-Push-Value and Polarized λ-Calculus (Abel et al. 2019)

A new entry in the Curry-Howard dictionary?

"**Cut-free proofs** have a good **subformula property**… but **sum elimination** complicates it."

"**Bidirectional typing** works well for **normal forms**… but **pattern-matching** complicates it."

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Thank you!

All formalised in Rocq, there's a paper at ITP '26