Gradualizing the Calculus of Inductive Constructions

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A bit of context

The Calculus of Inductive Constructions

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Gradual typing

- Mix static and dynamic typing
- Dynamic type ? in a static system
- Optimistic typing phase & runtime checks

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THEOREM PROVER

Agda

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Time has come for Gradual CIC!

Why gradual dependent types?

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Incremental development!

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filter (A : Type) (p : A → 𝔹) (n : ℕ) (l : Vect A n)
  : Vect A …
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head nat ? (filter nat even 0 [])
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head nat ? (filter nat even 3 [1;2;4]) ↦ 2
head nat ? (filter nat even 0 []) ↦ err
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filter (A : Type) (p : A → 𝔹) (n : ℕ) (l : Vect A n)
  : Vect A (count A p n l)


count A p n l := ?
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Effects

- Errors
- Divergence

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Indexed inductive types

- Vectors need special care
- Equality is currently out of our scope

An overview of gradual CIC

No-go: the fire triangle of graduality

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Our favorite properties

- Safety (Progress + Preservation)
- Normalization
- Conservativity (wrt. CIC)
- Graduality

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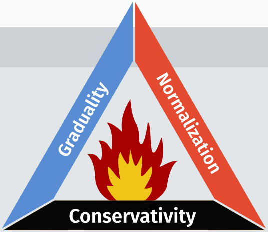

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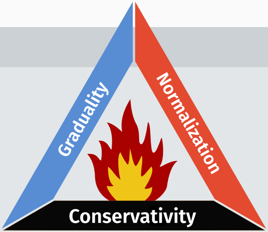
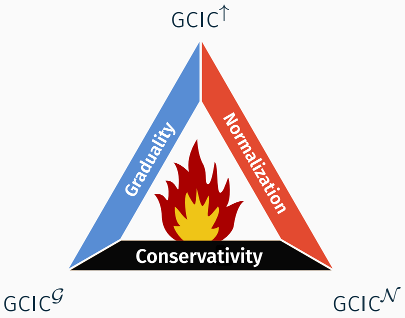


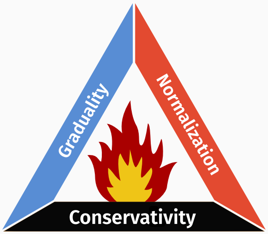
Safety + Conservativity + Graduality ⇒ Pure λ-calculus ⇒ Divergence


GCIC$^{\uparrow}$
Graduality
Normalization
Conservativity
GCIC$^{\mathcal{G}}$
GCIC$^{\mathcal{N}}$

Controlled by universe levels of Π types:

- at reduction (cast decomposition)
- at typing

$GCIC^{\uparrow}$



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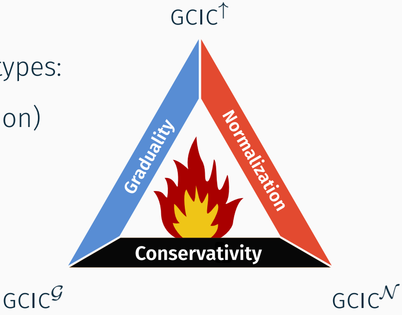

		GCIC$^{\mathcal{G}}$	GCIC$^{\mathcal{N}}$	GCIC$^{\uparrow}$
Reduction	Pure λ-calculus	×		
	Eager failure		×	×
Typing	CIC	×	×	
	Restricted			×

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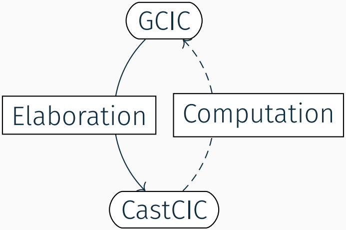

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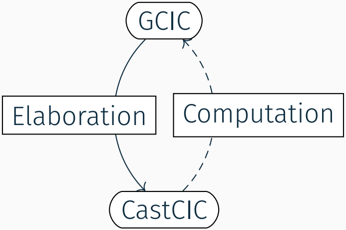

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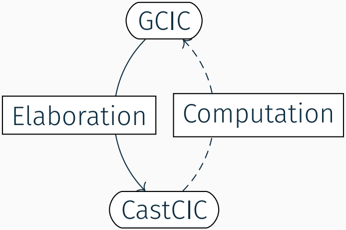

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- transitivity
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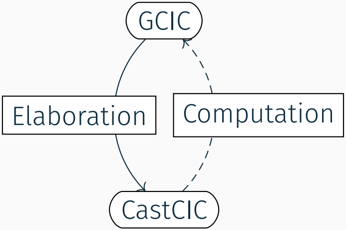

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- bidirectional typing
- type-based elaboration
- over approximation

New terms

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GCIC: $\cdots \mid ?$

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- $?_\square$ with casts as constructors/destructors:
  $\langle X \Leftarrow ?_\square \rangle\, \langle ?_\square \Leftarrow \mathbf{N} \rangle\, n \mapsto \langle X \Leftarrow \mathbf{N} \rangle\, n$

Theorems!

Type theory

Progress, preservation, normalization for castCIC.

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Proof: extension of *Sozeau et al., 2020* + translation back to CIC.

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Static gradual guarantee

Elaboration is monotone with respect to loss of precision.
$\Rightarrow$ *Syntactic* precision.

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Proof: on CIC consistency = conversion, using simulation again.

The semantic theorems

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Proof: build order models to interpret precision.

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That's it for today

Take-away

A quick summary

- No go: fire triangle of graduality
- GCIC: one system, three variants
- Type-based elaboration to CastCIC
- Safety and normalization
- Conservativity, static gradual guarantee and graduality
- Equality: still in progress

Thank you!