

GRADUALIZING THE CALCULUS OF INDUCTIVE CONSTRUCTIONS

M2 INTERNSHIP PRESENTATION

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INTRODUCTION

Gallinette team, LS2N, Nantes

Proof assistants (Coq), all over the place:

- Concrete implementation
- Proof assistant
- Theoretical background: type theory, effects, compilation...

Collaboration

Éric Tanter, Inria Paris/UCHile

Gradual typing

- Combine static and dynamic typing (Python 3.5, C#, Ruby, JavaScript...)
- Disciplined!

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Calculus of (Inductive) Constructions

- Powerful and expressive type theory/logics
- Old system, new features: errors

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But... Why?

Concretely Making programming in Coq easier/doable

Theoretically Good test for gradual typing and exceptional type theory

Introduction

1. Gradual Typing

Some Intuition

Gradually and Statically Typed λ -Calculus

2. Calculus of Inductive Constructions

3. Gradual CIC

What's Hard?

Overview of our solution

Perspectives

GRADUAL TYPING

What is the behaviour of $(\lambda x.S x)$ true?

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Static typing

Error at compile time

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$$\not\vdash (\lambda x : \mathbf{B}.S x) \text{ true}$$

Untyped λ -calculus

No error at compile time

$$\vdash (\lambda x.S x) \text{ true} : ?$$

Untrapped error

$$(\lambda x.S x) \text{ true} \mapsto S \text{ true} \mapsto \lambda x n y.n \text{ true } x \qquad (\lambda x.S x) \text{ true} \mapsto S \text{ true} \not\mapsto$$

TRAPPED AND UNTRAPPED ERRORS

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Trapped error

$(\lambda x.S x)$ true \mapsto S true \mapsto raise

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Mixing static & dynamic disciplines

With a safe head function:

$v : \text{Vect } 2 \vdash \text{head } v$ typechecks

$v : \text{Vect } ? \vdash \text{head } v$ dynamic check of the actual type of v

$v : \mathbf{N} \not\vdash \text{head } v$ type error

Mixing static & dynamic disciplines

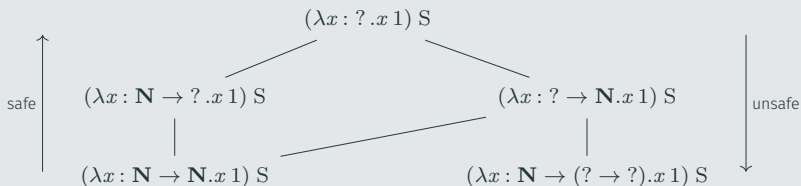
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Evolving types



Syntax and typing

$$T := \mathbf{B} \mid \mathbf{N} \mid T \rightarrow T$$

$$t := x \mid \underline{n} \mid \underline{b} \mid \lambda x : T. t \mid t t \mid t + t \mid \text{if } t \text{ then } t \text{ else } t$$

$$\frac{\Gamma, x : T_1 \vdash t : T_2}{\Gamma \vdash \lambda x : T_1. t : T_1 \rightarrow T_2}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2 \quad \Gamma \vdash t_3 : T_3 \quad T_1 = \mathbf{B} \quad T_2 = T \quad T_3 = T}{\Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : T}$$

...

Updated system

$$T := \mathbf{B} \mid \mathbf{N} \mid T \rightarrow T \mid ?$$

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...

Consistency

$$? \sim (\mathbf{N} \rightarrow ?) \quad \mathbf{N} \not\sim \mathbf{B} \quad (\mathbf{N} \rightarrow ?) \sim (? \rightarrow \mathbf{B}) \quad ((? \rightarrow ?) \rightarrow ?) \not\sim (\mathbf{N} \rightarrow ?)$$

Subject reduction is broken

$$\begin{aligned} &\vdash (\lambda x : ?.x + 1) \text{true} : \mathbf{N} \\ (\lambda x : ?.x + 1) \text{true} &\mapsto \text{true} + 1 \\ &\not\vdash \text{true} + 1 \end{aligned}$$

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Cast calculus

$$\vdash (\lambda x : ?.x + 1) \text{ true} \rightsquigarrow (\lambda x : ?.(cast_{?,\mathbf{N}} x) + 1) (cast_{\mathbf{B},?} \text{true})$$

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Cast calculus

$$\begin{aligned} &\vdash (\lambda x : ?.x + 1) \text{ true} \rightsquigarrow (\lambda x : ?.(cast_{?,\mathbf{N}} x) + 1) (cast_{\mathbf{B},?} \text{true}) \\ &(\lambda x : ?.(cast_{?,\mathbf{N}} x) + 1) (cast_{\mathbf{B},?} \text{true}) \mapsto (cast_{?,\mathbf{N}}(cast_{\mathbf{B},?} \text{true})) + 1 \\ &(cast_{?,\mathbf{N}}(cast_{\mathbf{B},?} \text{true})) + 1 \mapsto (cast_{\mathbf{B},\mathbf{N}} \text{true}) + 1 \mapsto \text{raise} \end{aligned}$$

CALCULUS OF INDUCTIVE CONSTRUCTIONS

- higher order, dependently typed extension of λ -calculus

$$\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash \Pi x : A. B : \square_i}{\Gamma \vdash \lambda x : A. t : \Pi x : A. B}$$

- with a conversion rule

$$\frac{\Gamma \vdash t : A \quad A \equiv B \quad \Gamma \vdash B : \square_i}{\Gamma \vdash t : B}$$

New base types, with three components:

- type (family)

$$\mathbf{Vect} : \Pi A : \square, n : \mathbf{N}.\square$$

- constructors

$$\mathbf{nil} : \Pi A : \square.\mathbf{Vect} A 0 \quad \mathbf{cons} : \Pi A : \square, n : \mathbf{N}, a : A, v : \mathbf{Vect} A n.\mathbf{Vect} A (S n)$$

- recursion/induction principle

$$\begin{aligned} \mathbf{rec}_{\mathbf{Vect}} : \Pi A : \square, P : (\Pi n : \mathbf{N}.\mathbf{Vect} A n \rightarrow \square). & (P 0 (\mathbf{nil} A)) \rightarrow \\ (\Pi n : \mathbf{N}, a : A, v : \mathbf{Vect} A n.P n v \rightarrow P (S n) (\mathbf{cons} A n a v)) \rightarrow & \\ \Pi n : \mathbf{N}, v : \mathbf{Vect} A n.P n v & \end{aligned}$$

And reduction rules

$$\begin{aligned} \mathbf{rec}_{\mathbf{Vect}} A P t_{\mathbf{nil}} t_{\mathbf{cons}} (\mathbf{nil} A) & \equiv t_{\mathbf{nil}} \\ \mathbf{rec}_{\mathbf{Vect}} A P t_{\mathbf{nil}} t_{\mathbf{cons}} (\mathbf{cons} A n a v) & \equiv t_{\mathbf{cons}} n a v (\mathbf{rec}_{\mathbf{Vect}} A P t_{\mathbf{nil}} t_{\mathbf{cons}} v) \end{aligned}$$

GRADUAL CIC

- in CIC, computation happens during typing :
$$\frac{\Gamma \vdash t : A \quad A \equiv B \quad \Gamma \vdash B : \square_i}{\Gamma \vdash t : B}$$
- dependent types: ? is everywhere
- natural consistency is HO unification:
Consistency definition \leftarrow/\rightarrow Decidable typing
- errors: tricky in CIC
- cast definition: what about inductive types? $\text{cast}_{\text{Id}_A x x', \text{Id}_A y y'}$?

Computation at typing

Separate conversions:

typing time simple and good enough: ? is just a constant

runtime fancy: errors, casting, etc.

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Undecidability of ideal consistency

- approximation
- design choice, with axiomatic approach
- open for further investigation

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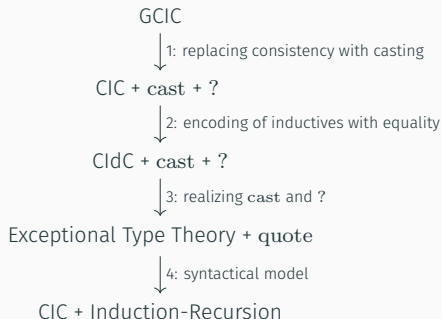
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Errors, type recursion

Syntactical models, brand new or heavy tweaking of existing ideas



PERSPECTIVES

Playing around

Aim: sanity check, testing...

- reduction of Ω and symptomatic terms
- handling of indexes: Vect A n
- compare consistency choices
- ideally, implementation

Ongoing work we could benefit from

- errors & effects in CIC
- quoting mechanism for Coq

THANK YOU!

QUESTIONS?

REDUCTION OF Ω

We write $?_i$ for $? \square_i$. We first get

$$\vdash \Omega : ?_{i+1} \rightsquigarrow (\lambda x : ?_{i+1} . (\text{cast}_{?_{i+1}, ?_{i+1} \rightarrow ?_{i+1}} x) x) (\text{cast}_{?_i \rightarrow ?_i, ?_{i+1}} (\lambda x : ?_i . \text{cast}_{?_i, ?_i \rightarrow ?_i} x x))$$

For readability, we set

$$\delta_i := \lambda x : ?_i . (\text{cast}_{?_i, ?_i \rightarrow ?_i} x) x$$

and

$$\Omega_i := \delta_{i+1} (\text{cast}_{?_i \rightarrow ?_i, ?_{i+1}} \delta_i)$$

The reduction now gives

$$\begin{aligned} \Omega_i &\mapsto^* (\text{cast}_{?_{i+1}, ?_{i+1} \rightarrow ?_{i+1}} \text{cast}_{?_i \rightarrow ?_i, ?_{i+1}} \delta_i) (\text{cast}_{?_i \rightarrow ?_i, ?_{i+1}} \delta_i) \\ &\mapsto^* (\text{cast}_{?_i \rightarrow ?_i, ?_{i+1} \rightarrow ?_{i+1}} \delta_i) (\text{cast}_{?_i \rightarrow ?_i, ?_{i+1}} \delta_i) \\ &\mapsto^* (\text{cast}_{?_i \rightarrow ?_i, ?_{i+1} \rightarrow ?_{i+1}} (\lambda x : ?_i . (\text{cast}_{?_i, ?_i \rightarrow ?_i} x) x)) (\text{cast}_{?_i \rightarrow ?_i, ?_{i+1}} \delta_i) \\ &\mapsto^* (\lambda x : ?_{i+1} . \text{cast}_{?_i, ?_{i+1}} ((\text{cast}_{?_i, ?_i \rightarrow ?_i} \text{cast}_{?_{i+1}, ?_i} x) \text{cast}_{?_{i+1}, ?_i} x)) (\dots) \\ &\mapsto^* (\lambda x : ?_{i+1} . \text{cast}_{?_i, ?_{i+1}} ((\text{cast}_{?_i, ?_i \rightarrow ?_i} (? \ ?_i)) (? \ ?_i))) (\text{cast}_{?_i \rightarrow ?_i, ?_{i+1}} \delta_i) \\ &\mapsto^* ? \ (?_{i+1}) \end{aligned}$$