Definitional Functoriality for Dependent (Sub)Types

Théo Laurent, **Meven Lennon-Bertrand**, Kenji Maillard

π^3 Seminar — December 11th 2023

Dependent types and subtyping

What users™ want:

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Too strict: $\dfrac{A' \preccurlyeq A \qquad B' \preccurlyeq B}{A \to B \preccurlyeq A' \to B'}$ is *not* set-theoretic inclusion...

What semanticists want you to do:

$$\textsc{Coe}\ \frac{\Gamma \vdash_{\mathrm{coe}} t : A \qquad \Gamma \vdash_{\mathrm{coe}} A \preccurlyeq A'}{\Gamma \vdash_{\mathrm{coe}} \mathrm{coe}_{A,A'}\, t : A'}$$

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Nicer semantics: $\left[\!\left[\mathrm{coe}_{A,A'}\right]\!\right]$ = "well-chosen functions"

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This is the work of a compiler!

Let's just elaborate, then:

$$\textsc{Sub}\ \frac{\Gamma \vdash_{\mathrm{sub}} t : A \qquad \Gamma \vdash_{\mathrm{sub}} A \preccurlyeq A'}{\Gamma \vdash_{\mathrm{sub}} t : A'} \quad \rightsquigarrow \quad \textsc{Coe}\ \frac{\tilde{\Gamma} \vdash_{\mathrm{coe}} \tilde{t} : \tilde{A} \qquad \tilde{\Gamma} \vdash_{\mathrm{coe}} \tilde{A} \preccurlyeq \tilde{A}'}{\tilde{\Gamma} \vdash_{\mathrm{coe}} \mathrm{coe}_{\tilde{A},\tilde{A}'}\, \tilde{t} : \tilde{A}'}$$

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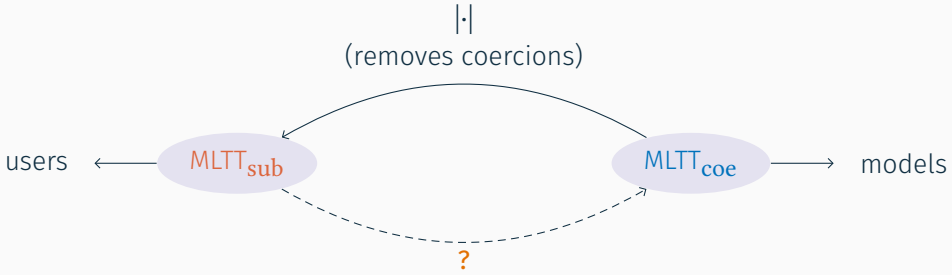
Better coherence: we should always have $\tilde{t} \cong \tilde{t}'$ for two different elaborations of t.

Now we have a goal



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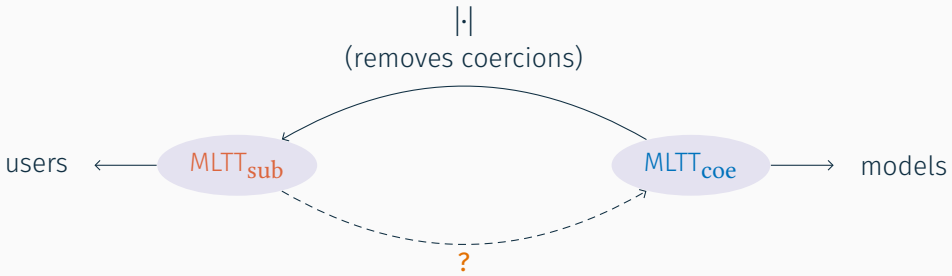

? = the compilation we want

Only well-defined if a lot of equations hold

In particular, $|t| = |u| \Rightarrow \Gamma \vdash_{\text{coe}} t \cong u : A$ (provided $\Gamma \vdash_{\text{coe}} t, u : A$)

Coercions are maps

What is a reasonable computational behaviour for coercions?

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(\mathrm{coe}_{A\to B,A'\to B'}\ f)\ u &\cong \mathrm{coe}_{B,B'}(f\ (\mathrm{coe}_{A',A}\ u)) \\
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These are map operations! We better understand them before going further.

Definitional functoriality

Definitional functor laws

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$$\textsc{MapComp}\ \frac{\Gamma \vdash_{\mathrm{map}} X, Y, Z : \mathrm{dom}(F) \quad \Gamma \vdash_{\mathrm{map}} g : \mathrm{hom}_F(X, Y) \quad \Gamma \vdash_{\mathrm{map}} f : \mathrm{hom}_F(Y, Z) \quad \Gamma \vdash_{\mathrm{map}} t : F\,X}{\Gamma \vdash_{\mathrm{map}} \mathrm{map}_F\ f\,(\mathrm{map}_F\, g\, t) \cong \mathrm{map}_F(f \circ^F g)\, t : F\,Z}$$

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+ congruences, specific laws for each F

$$\frac{\Gamma \vdash_{\mathrm{map}} (f, g) : \mathrm{hom}_\Pi((A, B), (A', B')) \qquad \Gamma \vdash_{\mathrm{map}} h : \Pi\, x{:}\, A.B \qquad \Gamma \vdash_{\mathrm{map}} a' : A'}{\Gamma \vdash_{\mathrm{map}} \mathrm{map}_\Pi\,(f, g)\, h\, a' \cong g\,(h\,(f\,a')) : B'[a']}$$

New equations for neutrals

This is *not* vanilla MLTT, where

$$\mathrm{map}_{\mathrm{List}}\ f\ (\mathrm{map}_{\mathrm{List}}\ g\ x) \not\equiv \mathrm{map}_{\mathrm{List}}(f \circ g)\ x$$

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In general, problems with neutrals at positive types (= without η)

Can we add these equations in?

arXiv:1304.0809v3 [cs.PL] 17 Jun 2013

New Equations for Neutral Terms

A Sound and Complete Decision Procedure, Formalized

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Abstract

The definitional equality of an intensional type theory is its test of type compatibility. Today's systems rely on ordinary evaluation semantics to compare expressions in types, frustrating users with type errors arising when evaluation fails to identify two 'obviously' equal terms. If only the machine could decide a richer theory! We propose a way to decide theories which supplement evaluation with 'ν-rules', rearranging the neutral parts of normal forms, and report a successful initial experiment.

We study a simple λ-calculus with primitive fold, map and append operations on lists and develop in Agda a sound and complete decision procedure for an equational theory enriched with monoid, functor and fusion laws.

Keywords Normalization by Evaluation, Logical Relations, Simply-Typed Lambda Calculus, Map Fusion

1. Introduction

The programmer working in intensional type theory is no stranger to 'obviously true' equations she wishes held *definitionally* for her program to typecheck without having to chase down ill-typed terms and brutally coerce them. In this article, we present one way to relax definitional equality, thus accommodating some of her longings. We distinguish three types of fundamental relations between terms.

The first denotes computational rules: it is untyped, *oriented* and denoted by $\rightsquigarrow$ in its one step version or $\rightsquigarrow^*$ when the reflexive transitive congruence closure is considered. In Table 1, we introduce a few such rules which correspond to the equations the programmer writes to define functions. They are referred to as δ (for *definitions*) and ι (for pattern-matching on *inductive* data) rules and hold computationally just like the more common β-rule.

The second is the judgmental equality ($\equiv$): it is typed, tractable

```
map : (a → b) → list a → list b
map f []        ⇝ []
map f (x :: xs) ⇝ f x :: map f xs

(++) : list a → list a → list a
[]      ++ ys ⇝ ys
x :: xs ++ ys ⇝ x :: (xs ++ ys)

fold : (a → b → b) → b → list a → b
fold c n []        ⇝ n
fold c n (x :: xs) ⇝ c x (fold c n xs)
```

Table 1. $\delta\iota$-rules - computational

```
Γ ⊢ f ≡ λ x. f x      : a → b
Γ ⊢ p ≡ (π₁ p , π₂ p) : a * b
Γ ⊢ u ≡ ()            : 1
```

Table 2. η-rules - canonicity

fied judgmentally. Table 3 shows a kit for building computationally inert *neutral* terms growing layers of thwarted progress around a variable which we dub the 'nut', together with some equations on neutral terms which hold only propositionally – until now. This paper shows how to extend the judgmental equality with these new 'ν-rules'. We gain, for example, that `map swap . map swap` $\equiv$ `id`, where `swap` swaps the elements of a pair.

```
⊠ □ a   π₁ □   π₂ □   □ ++ ys   map f □   fold n c □
```

DRAFT

New Equations for Neutral Terms

A Sound and Complete Decision Procedure, Formalized

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Decidability of Conversion for Type Theory in Type Theory

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JOAKIM ÖHMAN, IMDEA Software Institute, Spain
ANDREA VEZZOSI, Chalmers University of Technology, Sweden

Type theory should be able to handle its own meta-theory, both to justify its foundational claims and to obtain a verified implementation. At the core of a type checker for intensional type theory lies an algorithm to check equality of types, or in other words, to check whether two types are convertible. We have formalized in Agda a practical conversion checking algorithm for a dependent type theory with one universe à la Russell, natural numbers, and η-equality for Π types. We prove the algorithm correct via a Kripke logical relation parameterized by a suitable notion of equivalence of terms. We then instantiate the parameterized fundamental lemma twice: once to obtain canonicity and injectivity of type formers, and once again to prove the completeness of the algorithm. Our proof relies on inductive-recursive definitions, but not on the uniqueness of identity proofs. Thus, it is valid in variants of intensional Martin-Löf Type Theory as long as they support induction-recursion, for instance, Extensional, Observational, or Homotopy Type Theory.

CCS Concepts: • **Theory of computation → Type theory**; *Proof theory*;

Martin-Löf *à la* Coq

Jww. A. Adjedj, K. Maillard, P-M. Pédrot and L. Pujet

Martin-Löf logical framework

$$\vdash \Gamma \qquad \Gamma \vdash A \qquad \Gamma \vdash A \cong B \qquad \Gamma \vdash t : A \qquad \Gamma \vdash t \cong u : A$$

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$$\textsc{Refl}\ \frac{\Gamma \vdash t : A}{\Gamma \vdash t \cong t : A} \qquad \textsc{Sym}\ \frac{\Gamma \vdash t \cong u : A}{\Gamma \vdash u \cong t : A} \qquad \textsc{Trans}\ \frac{\Gamma \vdash t \cong u : A \qquad \Gamma \vdash u \cong v : A}{\Gamma \vdash t \cong v : A}$$

Martin-Löf logical framework

$$\vdash \Gamma \qquad \Gamma \vdash A \qquad \Gamma \vdash A \cong B \qquad \Gamma \vdash t : A \qquad \Gamma \vdash t \cong u : A$$

+ type formers ($\mathbf{Type}, \Pi, \Sigma, x =_A y, \mathrm{W} \ldots$)

$$\textsc{AppCong}\ \frac{\Gamma \vdash t \cong t' : \Pi\, x{:}\, A.B \qquad \Gamma \vdash u \cong u' : A}{\Gamma \vdash t\ u \cong t'\ u' : B[u]}$$

$$\beta\textsc{Fun}\ \frac{\Gamma \vdash A \qquad \Gamma, x{:}\, A \vdash B \qquad \Gamma, x{:}\, A \vdash t : B \qquad \Gamma \vdash u : A}{\Gamma \vdash (\lambda\, x{:}\, A.t)\, u \cong t[u] : B[u]} \qquad \eta\textsc{Fun}\ \frac{\Gamma \vdash f : \Pi\, x{:}\, A.B}{\Gamma \vdash f \cong \lambda x{:}\, A.f\ x : \Pi\, x{:}\, A.B}$$

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$$\vdash \Gamma \qquad \Gamma \vdash A \qquad \Gamma \vdash A \cong B \qquad \Gamma \vdash t : A \qquad \Gamma \vdash t \cong u : A$$

$$\textsc{Trans}\ \frac{\Gamma \vdash t \cong u : A \qquad \Gamma \vdash u \cong v : A}{\Gamma \vdash t \cong v : A} \qquad \textsc{Conv}\ \frac{\Gamma \vdash t : A \qquad \Gamma \vdash A \cong B}{\Gamma \vdash t : B}$$

+ type formers ($\mathrm{Type}, \Pi, \Sigma, x =_A y, \mathrm{W} \ldots$)

Derivations are not unique!

Declarative typing

Freestanding conversion rule

$$\frac{\Gamma \vdash^{\mathrm{de}} t : A \qquad \Gamma \vdash^{\mathrm{de}} A \cong B}{\Gamma \vdash^{\mathrm{de}} t : B}$$

Algorithmic typing (bidirectional)

Mode-constrained conversion

$$\frac{\Gamma \vdash^{\mathrm{al}} t \triangleright A \qquad \Gamma \vdash^{\mathrm{al}} A \cong B}{\Gamma \vdash^{\mathrm{al}} t \triangleleft B}$$

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Conversion mixes arbitrarily:

- Computation steps (β),
- Extensionality steps (η)
- Congruences,
- Transitivity, symmetry and reflexivity.

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Term/type-directed conversion alternating:

- Reduction to weak-head normal form,
- type-directed extensionality rules
- congruences

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Gives decidability of conversion/typing... and coherence! (More on this later)

The logical relation (Abel et al.)

A (proof-relevant) predicate $\Gamma \Vdash A$ characterizing types by their weak head normal form.

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For $\mathcal{A} :: \Gamma \Vdash A$, 3 predicates:

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Natural numbers:

$$\frac{\Gamma \vdash T \rightsquigarrow^{\star} \mathbf{N}}{\Gamma \Vdash T}$$

+ typing side-conditions

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- mutual definition of $\Gamma \Vdash A$ and $\Gamma \Vdash t : A$
- reducibility at the universe $\Gamma \Vdash A : \mathrm{Type}$ is basically $\Gamma \Vdash A$...

Flirting with logical limits

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(Small) induction-recursion + stratified definitions

Properties of the logical relation

- Escape: $\Gamma \Vdash A \Rightarrow \Gamma \vdash A$
- Irrelevance
- Equivalence: reflexivity, symmetry, transitivity
- Stability by weakening
- Neutral reflection
- Closure by anti-reduction

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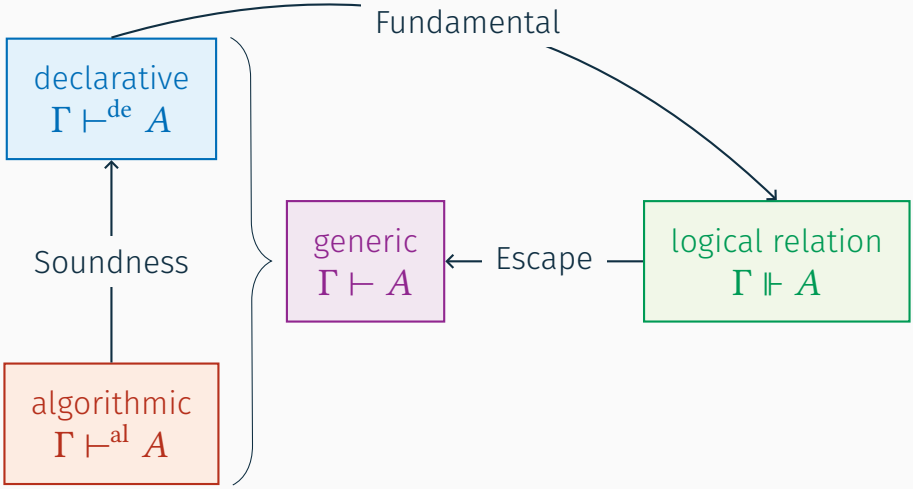
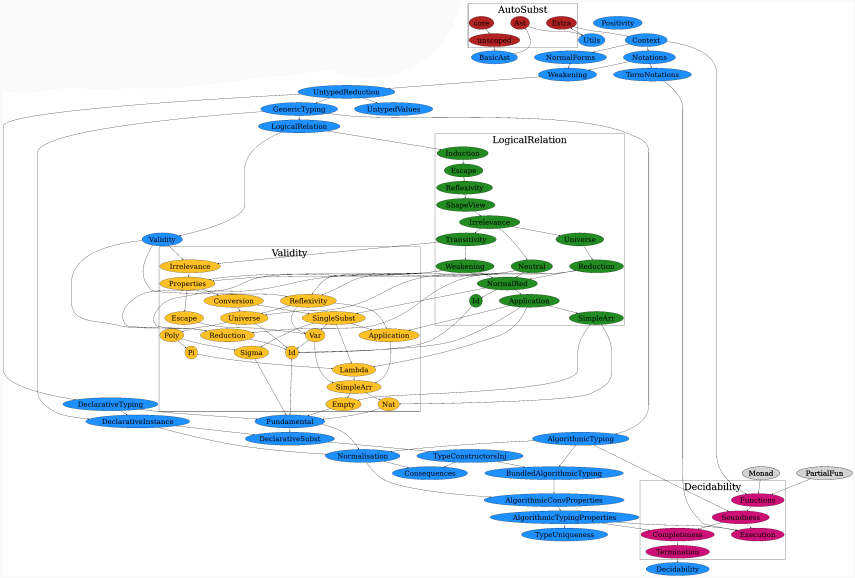
- Closure by anti-reduction

Fundamental lemma: if $\Gamma \vdash^{\mathrm{de}} t : A$ then $\mathcal{A} :: \Gamma \Vdash A$ and $\Gamma \Vdash_{\mathcal{A}} t : A$


Fundamental
declarative
$\Gamma \vdash^{\mathrm{de}} A$
Soundness
algorithmic
$\Gamma \vdash^{\mathrm{al}} A$
generic
$\Gamma \vdash A$
Escape
logical relation
$\Gamma \Vdash A$

All formalised!



Back to business

The whole story extends...

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To handle new equation for neutrals:

1. map fusion in reduction $\dfrac{\mathrm{neutral}\ n}{\mathrm{map}_{\mathrm{List}}\ f\ (\mathrm{map}_{\mathrm{List}}\ g\ n) \rightsquigarrow \mathrm{map}_{\mathrm{List}}(f \circ g)\ n}$

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List (⚙), W, Id, + ... (✒)

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- extend conversion to subtyping, reducible conversion to reducible subtyping
- $\mathbf{coe}_{A,B}\, t$ reduces A, B, then applies the relevant $\mathbf{map}$
- $\mathbf{coe}_{A,B}\, \mathbf{coe}_{A',B'}\, t$ is compacted if $A\ B\ A'\ B'$ are all neutral, or are positive types and t is neutral
- identity in comparison as before

Only the algorithmic system!

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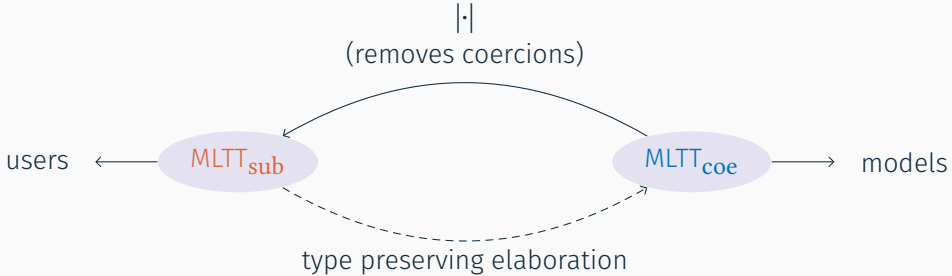
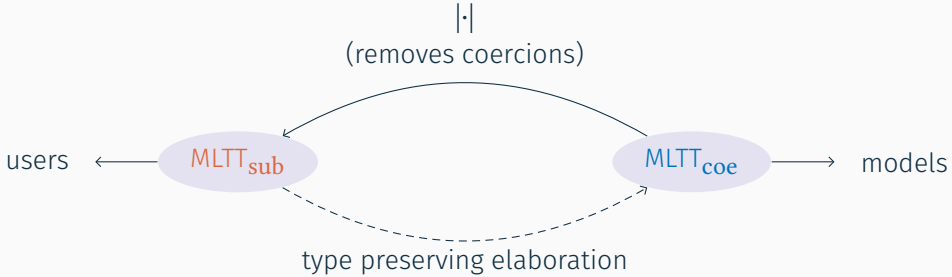


Much easier to show that elaboration preserves **algorithmic** typing.
Key lemma: coercions never block redexes.

Only the algorithmic system!



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For “free”: coherence up to conversion.

Wrapping up

The results

- Martin-Löf *à la* Coq: all meta-theory of MLTT, formalised in Coq (and there's more I have not told you about).

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To interpret subsumptive structural subtyping, you (only) need functoriality equations.

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- automation (proofs by reflection?)
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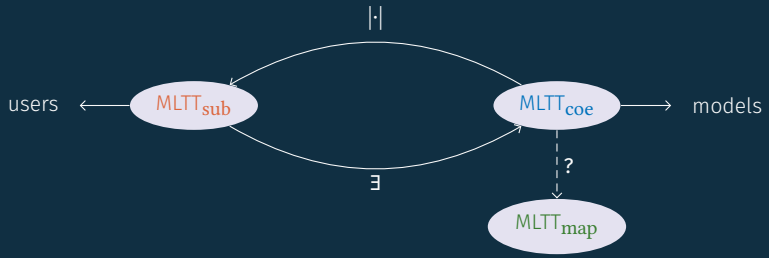
Functoriality and subtyping

- No formalisation of $\mathrm{MLTT}_{\mathrm{coe}}$/$\mathrm{MLTT}_{\mathrm{sub}}$
- No good story for elaborating $\mathrm{MLTT}_{\mathrm{coe}}$ to $\mathrm{MLTT}_{\mathrm{map}}$
- No general class of "good inductive types"



Thanks!

$$\Gamma \vdash_{\text{map}} \ \text{map}_F\, f : F\,X \to F\,Y \qquad \Gamma \vdash_{\text{map}} \ \text{map}_F\, \text{id}^F_X\ t \cong t : F\,X$$

$$\Gamma \vdash_{\text{map}} \ \text{map}_F\ f\,(\text{map}_F\, g\, t) \cong \text{map}_F(f \circ^F g)\, t : F\,Z$$